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Graduation Project report

Low SNR Characterization of the Channel Capacity of Generalized Fading Channels

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Hosting Organization:

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Dedications

To my beloved parents and siblings.

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Abstract

This paper aims to characterize the low signal-to-noise ratio (SNR) capacity of generalized fading channels. The work was performed during my graduation internship at King Abdullah University and Science and Technology. The multiple-input multiple-output (MIMO) Rayleigh fading channel and the single-input single-output (SISO) Log-normal shadowing channel are successively considered. General expressions of the ergodic capacities under different assumptions on channel state information (CSI) availability at the transmitter are presented. Then asymptotic analysis are made to derive low and high SNR asymptotic expressions of the capacities. The capacity of those channels has been shown to be affected by CSI availability at the transmitter only at low SNR. The tremendous capacity gain with multiple antennas was also retrieved and shown to decrease with the SNR. Simple on-off transmission schemes were then constructed and shown to be asymptotically capacity-achieving. Those schemes require only one bit feedback from the receiver to the transmitter so that the transmitter does not need to perfectly track the channel, showing that the proposed transmission schemes are really interesting from a practical point of view.

Keywords: Channel capacity, Asymptotic analysis, MIMO channel, Rayleigh fading, Log-normal shadowing, On-off scheme, Channel State Information.

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List of Abbreviations

3GPP	3rd Generation Partnership Project
3GPP2	3GPP2
AMPS	Advanced Mobile Phone System
AWGN	Additive White Gaussian Noise
CDMA	Code Division Multiple Access
CSI	Channel State Information
CSI-R	Channel State Information at the receiver
CSI-T	Channel State Information at the transmitter
EDGE	Enhanced Data Rates for GSM Evolution
FDMA	Frequency Division Multiple Access
FM	Frequency Modulation
GPRS	General packet radio service
GPS	Global Positioning Service
GSM	Global System for Mobile
ITU	International Telecommunications Union
LMDS	Local Multipoint Distribution Systems
LOS	line-of-sight

MIMO	multiple-input multiple-output
MISO	multiple-input single-output
MMDS	Multipoint Microwave Distribution Systems
MMS	Multimedia Messaging Service
NTT	Nippon Telegraph and Telephone
OFDM	Orthogonal Frequency Division Multiplexing
PAN	Personal Area Network
PC	Personal Computer
PDF	Probability Density Function
QoS	Quality of Service
SISO	single-input single-output
SMS	Short Messaging Service
SNR	signal-to-noise ratio
TACS	Total Access Communication System
TDMA	Time Division Multiple Access
UMTS	Universal Mobile Telecommunications System
WLAN	Wireless Local Area Network

List of Symbols

$\mathrm{H}(x)$	Entropy of x
$\mathrm{I}(x)$	Information of x
$f \gtrsim g$	f is greater than g at asymptotically low/high SNR: $\lim\limits_{\mathrm{SNR}\to 0/+\infty} \frac{f(\mathrm{SNR})}{g(\mathrm{SNR})} \geq 1$
$f \approx g$	f is equal to g at asymptotically low/high SNR: $\lim\limits_{\mathrm{SNR}\to 0/+\infty} \frac{f(\mathrm{SNR})}{g(\mathrm{SNR})} = 1$
$\mathbb{E}(.)$	Mathematical expectation
$\Gamma(.)$	Upper Incomplete Gamma function.
$\gamma(.)$	Lower Incomplete Gamma function.
$\det(\mathrm{M})$	determinant of matrix M.
$\mathrm{Tr}(M)$	trace of matrix M.
erfc(.)	Error complementary function.
$p_\lambda(.)$	Probability density function of λ
$\|.\|$	L^2 norm of a vector
$(x)^+$	maximum between 0 and x
$()!$	factorial of integer n
$\binom{i}{j}$	Binomial coefficient, with $0 \leq j \leq i$
y	channel output
x	channel input
H	complex $r \times t$ MIMO channel matrix
λ	Eigenvalue of $HH^\dagger$
μ	water-filling level in chapter 3
r	number of receive antennas in MIMO channel
t	number of transmit antennas in MIMO channel
m	minimum between r and t
n	maximum between r and t
ξ	Constant equal to $\frac{10}{\log(10)}$

List of Illustrations

Introduction

Wireless communications are certainly the most rapidly evolving area of communications. Since the Global System for Mobile (GSM) era when access to mobile technology exploded all over the world, many wireless technologies are now part of our daily life providing us various services such as voice communications, Short Messaging Service (SMS), world wide web browsing, wireless connectivity in personal area networks, etc. Those technologies include but are not limited to Wifi, Wimax, Bluetooth, Zigbee, Near-Field communications, satellite communications, microwave transmission, etc. In parallel, new technologies have also been developed for wired communications (e.g. optical fiber) providing high speed wired data communications to users. Since most electronic devices (mobile phones, personal computers, ...) are now able to connect to networks wirelessly, users expect wireless communications to be as fast as wired communications. This need has urged researchers to find new methods and techniques to enhance wireless communications performances.

One of those techniques is the multiple-input multiple-output (MIMO) which has attracted a lot of interest because of the high increase of capacity promised by this technology in the early works from [1], [2]. Indeed the capacity has been shown to be multiplied by a certain factor depending on the number of antennas used while keeping the same energy efficiency. MIMO technology is based on the principle of taking advantage of diversity and this diversity gain can still be obtained if we use multiple-input single-output (MISO) or SIMO systems. In order to implement in real world this technology a lot of work has been done regarding the impact of realistic channel models ([3], [4], etc.) since the early works were rather based on idealistic assumptions. An important result of those researches is that availability of Channel State Information (CSI) either at the transmitter or at the receiver has a large effect on MIMO systems capacity. While this effect is well characterized at high signal-to-noise ratio (SNR), the low SNR regime has remained rather unexplored. However with new systems operating at low SNR (wideband communications, sensor networks), there is a need to study the behavior of MIMO systems in the low power regime.

It is in this context that I carried out my graduation project in the wireless communications lab of KAUST and this document is the report on the work done during this internship. The aim of the project was to characterize the capacity of generalized fading channels at low SNR with a focus on the effect of full CSI that is perfect CSI is available both at the transmitter and the receiver.

This report is organized as follows: the first chapter recalls the general background of the project with a focus on MIMO channels which have been considered throughout the internship. The second chapter deals with the case when perfect Channel State Information at the receiver (CSI-R); the general capacity expression is derived followed by asymptotic analysis at high and low SNR. The third chapter embeds the main contributions of the internship with the study MIMO channel capacity under Rayleigh fading with full CSI; the noisy Channel State Information at the transmitter (CSI-T) is also considered in this chapter. Chapter 4 presents the first results from extending previous results to SISO channel undergoing Log-normal shadowing. The 5th chapter presents the first practical outputs of our work in the construction of practical and asymptotically capacity-achieving on-off transmission schemes.

Chapter I

Background Review

I.1 Wireless Technologies

Since the use of smoke for long distance communications in the pre-industrial era, mankind has achieved great advancement in wireless communications technologies at a point that they are now part of our daily life. In fact we use many electronic devices implementing one or many of the current wireless technologies which vary according to their range, their connectivity (to other networks), their performances (data rate, Quality of Service (QoS), etc.) and the different services they provide.

- **Cellular Systems** are the most spread wireless technology. They first provided worldwide voice communication with small handsets. As the data rates available are getting higher, data services are also provided including SMS, Multimedia Messaging Service (MMS), video calls, internet, etc. The most recent technology is LTE-Advanced and the most advanced terminals are smartphones.

- **Wireless Local Area Networks (WLANs)** comprise technologies specified in IEEE 802.11 standards such as Wifi or HiperLAN, they are used to connect computers or any supported device locally. They also often constitute access points to the Internet.

- **Broadband Wireless Access** has the particularity to involve no mobility with fixed agents communicating wirelessly. They mainly deal with point-to-point communication (Microwave) used by operators to interconnect their base stations, point-to-multipoint systems such as Local Multipoint Distribution Systems (LMDS) used for IP Telephony, and Multipoint Microwave Distribution Systems (MMDS) used for TV broadcasting.

- **Low-cost Low-power radios** refer to short-distance (tenths of meters) wireless technologies such Bluetooth, Zigbee and Ultra Wideband Radio. They are mainly used indoor for home appliances and in Personal Area Networks (PANs) to transfer data between devices such as mobile telephones, hands-free headsets, Personal Computers (PCs).

- **Satellite Systems** have been around for a long time now and have become a must especially in TV broadcasting and Global Positioning Service (GPS). Able to cover large regions, satellite communications are also used for many other applications including deserving mobile communications to low population density regions.

- **Cordless phones** are only used to provide mobility inside premises by connecting a mobile handset to the fixed station which in turn is connected to the telephone line.

- **Paging systems** are also part of the wireless technologies although they are not used all over the world due to mobile predominance. They provide short messaging with simple terminals.

Those technologies do not compete with each other but rather complete each other, most devices nowadays support at least two different wireless technologies. Let us now detail the description of mobile communications in harmony with our topic.

I.2 Mobile Communications

The first mobile telephony systems was introduced in 1946 in the USA. This system lacked efficiency in terms of power consumption and spectrum. Then Bell Labs introduced the cellular concept which exploits the fact that radio signal is attenuated with distance, so sets of frequencies could be used in different cells far enough from each other at the same time. The first generation of cellular systems saw the day in the early 80s followed by many improvements (see Figure I.1) that has led us to nowadays widespread high data rates mobile communications.

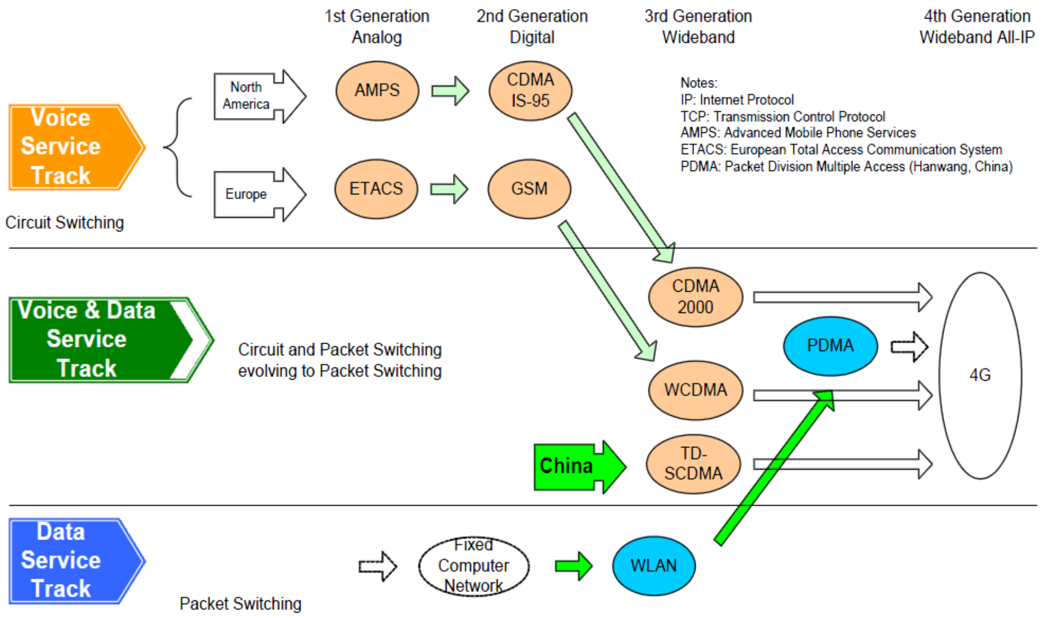

Figure I.1: Evolution of mobile radio communications, src:[5]

I.2.1 First Generation

The first generation of cellular systems was analog and used Frequency Modulation (FM) as modulation scheme and Frequency Division Multiple Access (FDMA) for the multiple access scheme. It was implemented as Advanced Mobile Phone System (AMPS) in the US, Total Access Communication System (TACS) in Europe and Nippon Telegraph and Telephone (NTT) in Japan. Those systems only provided voice communications. Later on, AMPS evolved into Digital-AMPS. Incompatibility between first generation systems especially in Europe led to the definition of second generation standards.

I.2.2 Second Generation

Digital communication was introduced in cellular systems in their second generation. The aim was to create global standards and introduce mobile data communications. However two standards competed for the world supremacy: GSM in Europe and IS-95 in the US. The difference between them resided in the multiple access technique; Time Division Multiple Access (TDMA) for GSM and Code Division Multiple Access (CDMA) for IS-95; and the frequency band they used. One can say that mobile phone really went popular with the second generation systems, operators introduced prepaid services and technology advancement allowed the manufacture of small handsets. On top of enhanced voice communications, SMS appeared. With the vulgarisation of Internet, mobile systems also needed to support data communications. GSM was then enhanced with General packet radio service (GPRS) and Enhanced Data Rates for GSM Evolution (EDGE) which not only brought the possibility to send multimedia through mobile networks but increased the data rate making high data rate applications such as video calling possible. All these improvements are classified in 2.5 and 2.75 mobile generations. Soon 3rd Generation systems were also implemented.

I.2.3 Third Generation

Looking for more harmony in the mobile telecommunications network worldwide, the International Telecommunications Union (ITU) defined specifications for the new third generations technologies. Two standards were established to assure continuity with existing 2G systems; Universal Mobile Telecommunications System (UMTS) by 3rd Generation Partnership Project (3GPP) for migration from GSM and CDMA2000 by 3GPP2 (3GPP2) for systems migrating from IS-95. CDMA was uniformly adopted as multiple access technique but frequency bands differed from one region to another due to political reasons. The peak data rates provided by these technologies are above 2 Mbps. Improvements were made to WCDMA in order to achieve even higher data rates first for the downlink with HSDPA allowing mobile phone applications involving downloading and then for uplink with HSUPA. Since mobile handsets become more powerful in terms of processing power and applications of mobile technology are diversifying to for example providing internet access to personal computers via usb dongles, there is still need for improvement in terms of data rates and inter-compatibility with other networks, bringing us to implementation of 4th generation standards.

I.2.4 Fourth Generation

This generation is already being implemented and brings IP technology to mobile networks with its all-IP network. There is no more circuit-switching so voice services are delivered through IP networks as VoIP. This generation introduces the real convergence of networks and also advanced wireless technologies such as MIMO.

I.3 Mobile Communication Challenges

Wireless channels are not as friendly as wired ones, even in the best case which is free space channel there is still noise to combat. In fact, many problems exist in wireless channels which have or are being adressed through the evolution of mobile technologies. We will now recall some of them which we believe are the most important.

I.3.1 Power Saving

Mobile communication systems have to be power conscious first because energy saving is part of nowadays goals, with the global warming and the depletion of oil resources, energy has become precious. The second reason is that mobile devices are energy limited mainly because of their size and users want their batteries to last as long as possible without recharging. Finally, companies want to reduce as much as possible operation costs in which energy cost is an important part.

I.3.2 Spectrum Management

Since the first generation cellular systems, spectrum management has been a concern leading to the introduction of cells and frequency reuse schemes. As the time goes on and the wireless technologies proliferate, spectrum has become more scarce so new ideas have to be found to overcome this problem. New multiple access techniques such TDMA, CDMA and Orthogonal Frequency Division Multiplexing (OFDM) have been adopted as solutions throughout the different generations. Now, new solutions are being introduced such as cognitive radio.

I.3.3 Multipath and Fading

As soon as we realize that the Additive White Gaussian Noise (AWGN) model is too optimistic, many propagation effects have to be considered. Those channel impairments can be divided into two types:

- *Large-scale fading* due to attenuation of a signal over distance and shadowing from large obstacles (buildings in urban environment).
- *Small-scale fading* due to multipath.

Many distributions are used to statistically model those fading for different environments.

- **The Rayleigh distribution** is used to model fast fading caused by local multipaths especially when there is no line-of-sight (LOS) between the transmitter and the receiver. The Rayleigh channel gain has the following distribution

$$f_h(t) = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} \tag{I.1}$$

  where σ^2 is the variance of h.

- When there is a LOS, the **Rician distribution** is used instead for small-scale fading. The Probability Density Function (PDF) is as follows

$$f_h(t) = \frac{t}{\sigma^2} e^{-\left(\frac{t^2+A^2}{2\sigma^2}\right)} I_0\left(\frac{At}{\sigma^2}\right) \tag{I.2}$$

  where A is the peak amplitude of the dominant signal, $I_0(.)$ the Bessel function of first kind and zero order, and σ is the standard deviation of h.

- The **Lognormal distribution** on the other hand models the shadowing effect of large obstacles on the signal. Its distribution is given by

$$f_h(t) = \frac{1}{\sqrt{2\pi}\sigma t} e^{-\frac{(\xi \log(t)-\mu)^2}{2\sigma^2}} \tag{I.3}$$

I.3.4 Interference Management

Another problem to overcome in mobile communications is interference starting from co-channel interference, other users interference, and even other technologies interference. Using orthogonal codes can help to manage some of this interference but since no code can be perfectly orthogonal the problem remains. Many interference cancellations techniques have been developed and mobile systems are designed with interference avoiding in mind.

I.4 Channel Capacity

I.4.1 Motivation

Since a signal is perturbed during his passage through a wireless channel there maybe a probability that it is not accurately detected at the receiver. One way to reduce this probability of error as much as we want is by simply reducing the rate of transmission. Thanks to Shannon, we now know that channel coding can also significantly improve the reliability of the communication channel without losing in terms of data rate. However Shannon also provided a certain limit to the maximal achievable rate for a certain channel above which there is no coding that can reduce as much as we want the probability of error. This maximal achievable rate is called the capacity. Its definition goes back to the information theory where it represents the maximum mutual information between the received signal and the transmitted one.

I.4.2 Definition

As soon as we wanted to rationally analyze communication systems, there was a need to mathematically define the term information. Shannon came up with a good definition based on certain properties that this tool should possess such as the information should not be negative, the information provided by a rare event should be greater than that of a common event. So for a random process X, the information $\mathrm{I}(X = x)$ of the event $X = x$ is defined as

$$\mathrm{I}(x) = -\log(p_X(x)) \tag{I.4}$$

Since X is a random process we may think of the average over all realisations x of $\mathrm{I}(x)$ which is called the entropy of X and noted $\mathrm{H}(X)$

$$\mathrm{H}(X) = \mathbb{E}_X[\mathrm{I}(X = x)] = \mathbb{E}_x[-\log(p_X(x))] \tag{I.5}$$

Now coming back to our framework, we have at least two random processes involved which are the tansmitted signal and the received one. We then need to mathematically define some relations between those two randoms processes in terms of information. From probability theory we already have the joint probabilty and the conditional probability between them. So the conditional entropy is defined as

$$\mathrm{H}(Y|X) = \mathbb{E}_{X,Y}[-\log(p_{X|Y}(x|y))] \tag{I.6}$$

and the joint entropy or mutual information is defined as

$$\mathrm{I}(X,Y) = \mathrm{H}(Y) - \mathrm{H}(Y|X) = \mathrm{H}(X) - \mathrm{H}(X|Y) \tag{I.7}$$

The mutual information between the transmitted signal and the received one represents the data rate over the transmission channel. So maximizing it yields us the channel capacity.

$$C = \max_{x,y} \mathrm{I}(x,y) \tag{I.8}$$

This maximization is mainly done by formatting the transmitted signal in a way such that the channel has no or little bad effect on it.

I.4.3 SISO Channel Capacity

AWGN Channel

The AWGN channel is one of the simplest transmission channel models often used as a start point for building more complex channel models and evaluating their performance. However it is a good channel model for free space communications. The output $y \in \mathbb{C}$ of an AWGN channel is given by

$$y = x + n \tag{I.9}$$

where $x \in \mathbb{C}$ is the transmitted signal with power P and $n \in \mathbb{C}$ is the Gaussian white noise of variance N_0. The capacity of this channel is given by

$$C_{\mathrm{AWGN}} = \frac{1}{2}\log(1 + \mathrm{SNR}) \tag{I.10}$$

where $\mathrm{SNR} = \frac{P}{N_0}$ is the signal-to-noise ratio.

Fading Channel

If we now consider the presence of fading in the previous AWGN channel model, we will have

$$y = hx + n \tag{I.11}$$

where h is the channel gain which can have any of the distributions mentioned in section I.3.3. The formula of the instantaneous capacity with constant transmit power remains the same as that of the AWGN channel with only the SNR depending on h as

$$\text{SNR} = \text{SNR}_{awgn}|h|^2 \tag{I.12}$$

Since h is a random process, averaging this instantaneous capacity over all realisations of the channel gives us the ergodic capacity defined as

$$C^0 = \mathbb{E}_{h^2}\left[\log(1 + \frac{P}{N_0}|h|^2)\right] \tag{I.13}$$

The constant transmit power scheme is used when the transmitter has no CSI. Otherwise the transmit power is adapted to the channel quality using the water-filling algorithm.

With the increased vulgarization of mobile communications, there has been an increased need for high data rates. This has led the scientific community to look for new transmission techniques that can significantly improve the the quality of wireless communications among which the MIMO technology.

I.5 MIMO Channels

I.5.1 Motivation

In order to increase data rates or reduce the error probability several methods can be used among which is to take advantage of diversity. Diversity means to mix information from different sources and extract more valuable or reliable information. Diversity can be considered over time where delayed versions of a signal are used to detect with more reliability the original signal, over frequency where the signal is sent on different carriers, and over space where multi-path is used as the source of diversity. An application of the latter is the use of multiple antennas at the transmitter and/or at the receiver which is called

MIMO. A tremendous interest has been devoted on those systems since the late 90s when pioneering works from Telatar [1] and Foschini [2] promised incredible increase of capacity for them compared to traditional SISO systems. In fact theoretical results predicted the capacity to be multiplied by the number of parallel channels (minimum between number of transmit antennas and number of receive antennas) available in those MIMO systems. However those optimistic promises are now nuanced with taking into account various imperfections affecting the channel such as fading, co-channel interference, etc. So a lot of study had to be done in order to see practical implementation of this promising technology which is already the case. MIMO technology is part of the latest releases of advanced wireless technologies such as WIFI(IEEE 802.11n), LTE, LTE-Advanced,WiMax,...

I.5.2 System Model

We consider that instead of one antenna at both sides of the channel, we have t transmit antennas and r receive antennas. The output $y \in \mathbb{C}^r$ is now written as

$$y = Hx + n \tag{I.14}$$

where $x \in \mathbb{C}^t$ is the transmitted signal, $n \in \mathbb{C}$ is complex Gaussian noise and H is a $r \times t$ complex matrix representing the channel gain. $H_{i,j}$ represents the channel gain affecting the path between the j^{th} transmit antenna and the i^{th} receive antenna. In an AWGN channel, $H_{i,j}$ is deterministic but in general it is a random variable whose distribution depends on the fading considered. For Rayleigh fading, $H_{i,j}$ is complex Gaussian with zero mean, independent real and imaginary parts each with variance 1/2.

I.5.3 Applications

MIMO technology is used in different ways in order to reduce interference, take advantage of diversity and increase data rates or decrease error probability. MIMO can be used for space-time transmit diversity by sending the same data through different antennas. This help reduce the error probability at the receiver by using diversity combining techniques. MIMO can also be used for spatial multiplexing where parallel streams of data are sent through different antennas. The multipath helps in this case to mitigate between the different streams. MIMO can also be considered in a multi-user scenario where the different

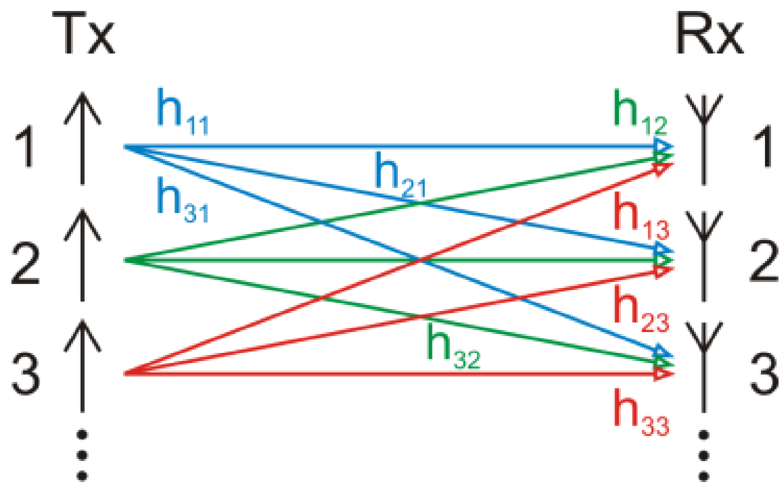

Figure I.2: MIMO channel overview, src:http://en.wikipedia.org/wiki/MIMO

mobile station connected to the network are considered separate antennas of the same entity, these scenarios refer to collaborative MIMO or MIMO-broadcasting. With its tremendous advantages, MIMO has been adopted by other technologies such as radar and even DSL applications to reduce cross-talk between cable binders.

Chapter II

MIMO channel capacity under Rayleigh fading with no CSI-T and perfect CSI-R

II.1 Introduction

The first analyses made on MIMO systems considered a time-invariant channel matrix [1]. Using singular value decomposition of the fixed channel matrix, the MIMO channel can be decomposed into $m = min(r, t)$ single-input single-output (SISO) sub-channels. This suggests that the capacity of a MIMO channel is m times that of a SISO channel with the same power constraint. This basic model was then replaced by more involved models taking into account variability over time and also randomness. In this chapter, we discuss the case when the complex random channel experiences Rayleigh fading but its instantaneous realization is known to receiver only. We first derive the general expression of the ergodic capacity of such channel and then using asymptotic analysis, we provide the low and high SNR regime behavior of that capacity.

II.2 General Expression

By noting Q the transmit signal covariance, the rate achieved in a particular channel state H is

$$\log\det\left(I_r + \frac{1}{N_0}HQH^\dagger\right) \tag{II.1}$$

where N_0 is the variance of the complex additive Gaussian noise. The ergodic rate for the same transmit covariance Q is then obtained by averaging over all realizations of H

$$\mathbb{E}_H\left[\log\det\left(I_r + \frac{1}{N_0}HQH^\dagger\right)\right] \tag{II.2}$$

Now to obtain the ergodic capacity, we have to maximize this quantity subject to the total transmit average power constraint $\mathrm{Tr}[Q] \leq P_{avg}$. The maximization variable here is the transmit signal covariance Q, which is basically affected by the channel coding at the transmitter. So the capacity is

$$C^1 = \max_{Q:\mathrm{Tr}[Q]\leq P_{avg}} \mathbb{E}_H\left[\log\det\left(I_r + \frac{1}{N_0}HQH^\dagger\right)\right] \tag{II.3}$$

Since the transmitter has no knowledge of the complex random matrix of the channel, and the entries of the channel matrix are i.i.d. (Rayleigh fading environment), which means that all transmit paths are statistically equivalent, it has been proven in the literature that distributing equally the transmit power among antennas is the optimal power allocation. The general expression of the capacity is then given by [1, Theo. 1] as

$$C^1 = \mathbb{E}_H\left[\log\det\left(I_r + \frac{\mathrm{SNR}}{t}HH^\dagger\right)\right] \tag{II.4}$$

where $\mathrm{SNR} = \frac{P_{avg}}{N_0}$ is the average signal-to-noise ratio, t the number of transmit antennas and r the number of receive antennas. The distribution of H being known, this expression can be transformed in a more closed-form by using the PDF of the unordered eigenvalues of the Wishart matrix $HH^\dagger$. The capacity is then given by [1, Eq. (7)]

$$C^1 = m\mathbb{E}_\lambda\left[\log\left(1 + \frac{\mathrm{SNR}}{t}\lambda\right)\right] \tag{II.5}$$

A tractable expression of the PDF of λ is given in [6, Eq. (42)] as

$$p_\lambda(\lambda) = \frac{1}{m}\sum_{i=0}^{m-1}\sum_{j=0}^{i}\sum_{l=0}^{2j} c(n,m,i,j,l)\lambda^{n-m+l}e^{-\lambda} \tag{II.6}$$

where $c(n,m,i,j,l)$ is given by

$$c(n,m,i,j,l) = \frac{(-1)^l(2j)!}{2^{(2i-l)}j!l!(n-m+j)!}\binom{2i-2j}{i-j}\binom{2n-2m+2j}{2j-l} \tag{II.7}$$

This expression allows an easy computation of the CSI-R capacity but it does not provide an analytical insight of the effect of the different parameters (SNR,r,t) on that capacity. That is why it is useful to also provide simple analytical asymptotic expressions of the capacity describing its behavior at high or low SNR.

II.3 Asymptotic Analysis

The asymptotic analysis here consist in finding asymptotic expression of the capacity in (II.5) by letting SNR $\to 0$ or SNR $\to \infty$. This technique is useful in providing interesting insights on the behavior of capacity at different SNR ranges. We provide here those asymptotic expressions and numerical results comparing them with the exact capacity.

II.3.1 High SNR Regime

We consider here that SNR $\to +\infty$. In this case, $1 + \frac{\text{SNR}}{t}\lambda \approx \frac{\text{SNR}}{t}\lambda$. Applying this result in (II.5), the high SNR capacity of the Rayleigh fading MIMO channel can be written as

$$C^1_\infty \approx m\mathbb{E}_\lambda\left[\log\left(\frac{\text{SNR}}{t}\lambda\right)\right] \tag{II.8}$$

$$\approx m\log(\text{SNR}) + m\mathbb{E}_\lambda[\log(\lambda)] - m\log(t) \tag{II.9}$$

$$\approx m\log(\text{SNR}) \tag{II.10}$$

Recalling that the SISO channel capacity scale essentially as log(SNR), we clearly see the capacity gain of magnitude m obtained with the use of multiple antennas. We can also see that it is useless to have $r < t$ or $r > t$ that is different number of antennas at both sides of the channel. Another surprising insight is that at high SNR, to increase the number of antennas has more impact than increasing the transmit power. But one must remember that these conclusions are made in a context where the receiver knows perfectly the channel which is not always the case in practical systems.

II.3.2 Low SNR Regime

We consider here that SNR $\to 0$. In this case, we can say that $\log\left(1 + \frac{\text{SNR}}{t}\lambda\right) \approx \frac{\text{SNR}}{t}\lambda$. Injecting this result into (II.5), the low SNR capacity of the Rayleigh fading MIMO channel can be written as

$$C_0^1 \approx m\mathbb{E}_\lambda \left[\frac{\text{SNR}}{t}\lambda\right] \tag{II.11}$$

$$\approx m\left(\frac{\text{SNR}}{t}\right)\mathbb{E}_\lambda\left[\lambda\right] \tag{II.12}$$

$$\approx r\ \text{SNR} \tag{II.13}$$

where (II.13) follows from the fact that $\mathbb{E}_\lambda[\lambda] = n$ (see Appendix A) and $\frac{m\times n}{t} = r$. Here also, recalling that the SISO channel capacity scales as SNR in the low power regime, we can see that unlike in the high power regime, the capacity grows linearly with the number of receive antennas. This insight can help in the design of low power MIMO systems. Another difference with high power regime is that the capacity is linear in SNR which means that SNR increase in the low power regime has more impact on MIMO channels capacity with receiver CSI than in the high power regime.

II.3.3 Numerical Results

The figures II.3.3 and II.3.3 present MIMO channel capacities under Rayleigh fading and assuming perfect CSI only at the receiver for different antenna configurations. Figure II.3.3 presents the 2 transmit - 2 receive antennas MIMO capacity along with asymptotic expressions derived in (II.10) and (II.13), and also the SISO channel capacity. This figure clearly shows the tightness of the asymptotic expressions and the capacity gain obtained by only one supplementary antenna at both sides of the channel. As predicted the channel capacity is linear in the low SNR regime and logarithmic in the high SNR regime. Figure II.3.3 on the other hand presents the evolution of MIMO CSI-R capacity for different antenna configurations which are:

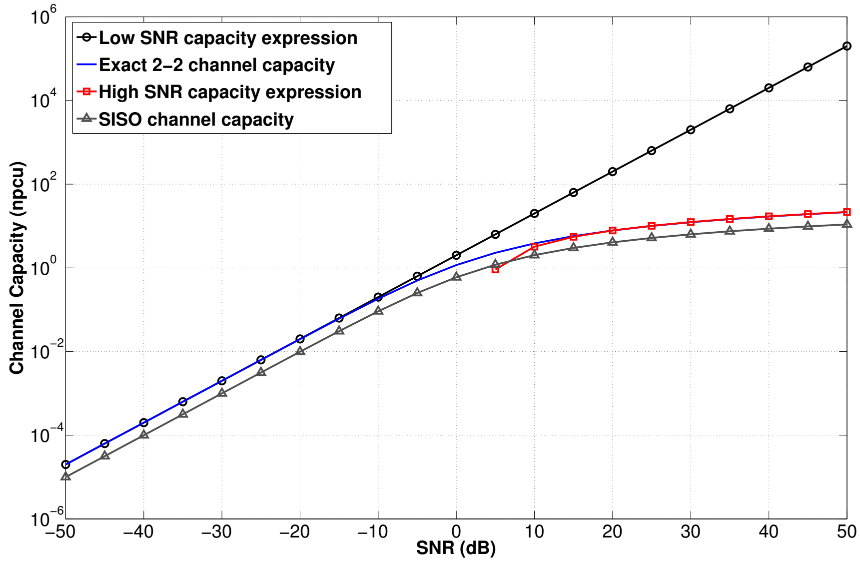

Figure II.1: 2 transmit and 2 receive antennas channel capacity with perfect CSI-R in nats per channel use (npcu) versus SNR in dB.

- 3 transmit antennas and 5 receive antennas MIMO channel (C1)
- 5 transmit antennas and 3 receive antennas MIMO channel (C2)
- 3 transmit antennas and 3 receive antennas MIMO channel (C3)
- SISO channel

Figure II.3.3 also demontrates the capacity gain provided by MIMO systems. Also as predicted before, C1,C2 and C3 capacities are almost indistinguishable in the high SNR region, indeed they have the same high SNR asymptotic capacity as shown in (II.10). However, the capacity of C1 becomes greater than those of C2 and C3 in the low power regime which can be predicted using (II.13) since C2 and C3 have only three receive antennas while C1 has five of them.

II.4 Conclusion

In this chapter we have presented the ergodic capacity of a Rayleigh fading MIMO channel assuming perfect knowledge of the channel at the receiver and perfect decoding also. Using asymptotic analysis, we have retrieved the fact that the MIMO capacity is m times that

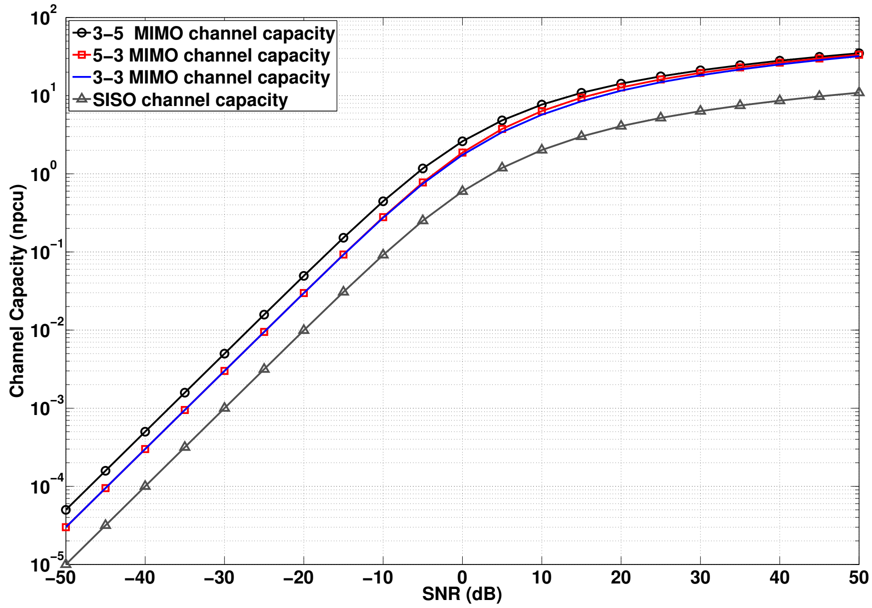

Figure II.2: MIMO channels capacities with perfect CSI-R in nats per channel use (npcu) versus SNR in dB.

of a SISO channel with m being the minimum between the number of transmit antennas and the number of receive antennas. Some plots were provided to numerically support the results. In the next chapter, we will tackle the case when the transmitter also has a perfect knowledge of the channel.

Chapter III

MIMO channel capacity under Rayleigh fading with perfect CSI-T and perfect CSI-R

III.1 Introduction

Even if the no CSI-T case treated in the previous chapter is more realistic than assuming instantaneous CSI availability at the transmitter, there are some cases where the transmitter is actually able to track to the channel gain. We can also imagine a scenario where there is feedback from the receiver to the transmitter so that both of them are able to track the channel gain. In this case the transmitter can adapt his transmitted signal in order to minimize the bad effects of the channel and this could enable the design of less complex detectors in the receiver. We can already predict intuitively that in such a scenario, the system will perform better than the previous case. In this chapter, we will study the capacity of MIMO channel undergoing Rayleigh fading assuming perfect CSI at both the transmitter and the receiver. As in the previous chapter, we will also study the asymptotic behavior of this capacity respectively at high and low SNR.

III.2 General Expression

In order to derive the full CSI ergodic capacity of a MIMO channel, we will first derive the capacity for a time-invariant MIMO channel which is also the instantaneous capacity in the ergodic case. We recall that the channel output $y \in \mathbb{C}^r$ is written as

$$y = Hx + n \tag{III.1}$$

where $x \in \mathbb{C}^t$ is the transmitted signal, $n \in \mathbb{C}$ is complex Gaussian noise and H is a $r \times t$ complex matrix representing the channel gain and is deterministic. Using the singular value decomposition theorem, we can write

$$y = U\Lambda V^\dagger x + n \tag{III.2}$$

where $U \in \mathbb{C}^{r\times r}$ and $V \in \mathbb{C}^{t\times t}$ are unitary matrices, $\Lambda \in \mathbb{C}^{r\times t}$ is a rectangular diagonal matrix with non-negative real numbers on the diagonal. If we pose $\tilde{y} = U^\dagger y$, $\tilde{x} = V^\dagger x$ and $\tilde{n} = U^\dagger n$; we will have

$$\tilde{y} = \Lambda\tilde{x} + \tilde{n} \tag{III.3}$$

Note that $\tilde{n}$ has the same distribution as n and $\|\tilde{x}\|^2 = \|x\|^2$, so the energy is conserved through this transformation. Denoting by $\sqrt{\lambda_i}, i = 1\cdots$ the eigenvalues of H where $m = min(r,t)$, the channel in (III.1) can be seen as m parallel SISO fading channels written as

$$\tilde{y}_i = \sqrt{\lambda_i}\tilde{x}_i + \tilde{n}_i \tag{III.4}$$

In order to maximize the mutual information between $\tilde{y}$ and $\tilde{x}$, we need to choose $\{\tilde{x}_i, i = 1\ldots m\}$ to be independent. Then from I.4.3, we can deduce that the capacity of this channel can be written as

$$C_{ins}^2 = \max_{\tilde{x},\|\tilde{x}\|^2=P} \sum_{i=1}^{m} \log\left(1 + P_i^*\lambda_i\right) \tag{III.5}$$

Solving this maximization problem (see [7, Example 5.2, p. 245]), yields us an optimal power profile P_i^* allocated to the eigen-mode i by the water-filling algorithm as

$$P_i^* = \left(\mu - \frac{1}{\lambda_i}\right)^+ \tag{III.6}$$

where μ is chosen to meet the power constraint $\sum_{i=1}^{m} P_i^* = SNR$ The capacity is the rewritten as

$$C_{ins}^2 = \sum_{i=1}^{m} \log\left(\mu\lambda_i\right)^+ \tag{III.7}$$

This result can be easily extended to ergodic case and noticing that λ_i are i.i.d. in the Rayleigh fading case, we get

$$C^2 = m\mathbb{E}_\lambda\left[\log\left(\mu\lambda\right)^+\right] \tag{III.8}$$

where μ is chosen to meet the average power constraint

$$\mathrm{SNR} = m\mathbb{E}_\lambda\left(\mu - \frac{1}{\lambda}\right)^+ \tag{III.9}$$

A PDF of unordered i.i.d. eigenvalues of the Wishart matrix $HH^\dagger$ is given in [1] as

$$p_\lambda(\lambda) = \frac{1}{m}\sum_{i=0}^{m-1}\frac{k!}{(n-m+k)!}\left[L_k^{n-m}(\lambda)\right]\lambda^{n-m}e^{-\lambda} \tag{III.10}$$

where $L_i^j(x)$ is the associated Laguerre polynomial of order k defined in [8, Eq. (8.972)]. This PDF involving the Laguerre polynomials is rather difficult to manipulate. Instead we used another expression of that PDF given in [6, eq. 42] as

$$p_\lambda(\lambda) = \frac{1}{m}\sum_{i=0}^{m-1}\sum_{j=0}^{i}\sum_{l=0}^{2j} C(n,m,i,j,l)\lambda^{n-m+l}e^{-\lambda} \tag{III.11}$$

where $C(n,m,i,j,l)$ is given by

$$C(n,m,i,j,l) = \frac{(-1)^l(2j)!}{2^{(2i-l)}j!l!(n-m+j)!}\binom{2i-2j}{i-j}\binom{2n-2m+2j}{2j-l} \tag{III.12}$$

(III.8) can be used to compute numerically to provide the capacity for any SNR. However it does not provide a means to analytically study the behavior of the capacity especially for certain SNR regimes. We provide hereafter asymptotic analysis at high and low SNR in order to better understand the impact of the different parameters on the capacity.

III.3 Asymptotic Analysis

The asymptotic analysis of the full CSI capacity of MIMO channel reveals to be more involved especially for the low SNR regime than the receiver CSI case. Before getting into the subject, we will first recall a result from [9] about the impact of SNR on μ. The function $G(.)$ is defined as

$$G(x) = m\mathbb{E}_\lambda \left(\frac{1}{x} - \frac{1}{\lambda} \right)^+ \tag{III.13}$$

Now, by using [9, Lemma 1], we can say that

$$\lim_{\text{SNR}\to 0} \mu = \lim_{\text{SNR}\to 0} G^{-1}(\text{SNR}) = 0 \tag{III.14}$$

So when considering that SNR $\to 0$, we can also say that $\mu \to 0$. On the other hand, we have

$$\begin{aligned} \text{SNR} &= m\mathbb{E}_\lambda \left(\mu - \frac{1}{\lambda} \right)^+ \\ &\leq m\mu \end{aligned}$$

So when SNR goes to ∞, so does μ.

III.3.1 High SNR Regime

We consider here that SNR $\to +\infty$. In this case, we have seen that μ also grows to infinity. Then, from (III.9) we can say that

$$\text{SNR} \approx m\mathbb{E}_\lambda \left[\left(\mu - \frac{1}{\lambda} \right) \right] \tag{III.15}$$

which gives us

$$\mu(\text{SNR}) \approx \frac{\text{SNR}}{m} + \mathbb{E}_\lambda \left(\frac{1}{\lambda} \right) \tag{III.16}$$

$$\approx \frac{\text{SNR}}{m} \tag{III.17}$$

The asymptotic expression of the capacity is then given by

$$\begin{aligned} C_\infty^2 &\approx m\mathbb{E}_\lambda \left[\log\left(\mu\lambda\right)\right] && \text{(III.18)} \\ &\approx m\log(\mu(\mathrm{SNR})) + m\mathbb{E}_\lambda\left[\log\left(\lambda\right)\right] && \text{(III.19)} \\ &\approx m\log(\mathrm{SNR}) && \text{(III.20)} \end{aligned}$$

This result is rather interesting as it clearly suggests that CSI at the transmitter has little effect on MIMO channel capacity at high SNR. This can help reduce complexity in high power MIMO systems. This results confirms the tremendous capacity gain (order of magnitude m) obtained with MIMO systems.

III.3.2 Low SNR Regime

As stated before, the low SNR asymptotic analysis of the full CSI MIMO capacity is more involved. We present the result as the following theorem and provide its proof.

Theorem. *The capacity of a MIMO channel undergoing Rayleigh fading as described in* (I.14) *with perfect CSI-T and CSI-R in the low SNR regime is given by*

$$\begin{aligned} C &\approx \begin{cases} -\alpha\ SNR\ W_0\left((SNR)^{\frac{1}{\alpha}}\right) & \text{if } \alpha < 0, \\ -SNR\log(SNR) & \text{if } \alpha = 0, \\ -\alpha\ SNR\ W_{-1}\left(-(SNR)^{\frac{1}{\alpha}}\right) & \text{if } \alpha > 0, \end{cases} && \text{(III.21)} \\ &\approx SNR\log(1/SNR) && \text{(III.22)} \end{aligned}$$

where $\alpha = n + m - 4$, $W_0(.)$ *and* $W_{-1}(.)$ *are the main and the lower branches of the Lambert-*W *function, respectively.*

Proof. First let us find the water-filling level μ that corresponds to optimal power allocation. From (III.9) and (III.11), we have

$$\mathrm{SNR} = \sum_{i=0}^{m-1}\sum_{j=0}^{i}\sum_{l=0}^{2j} C(n,m,i,j,l)\int_{\frac{1}{\mu}}^{\infty}\left(\mu - \frac{1}{\lambda}\right)\lambda^{n-m+l}e^{-\lambda}\,d\lambda,$$

The integral

$$I_1 = \int_{\frac{1}{\mu}}^{\infty}\left(\mu - \frac{1}{\lambda}\right)\lambda^{n-m+l}e^{-\lambda}\,d\lambda \qquad \text{(III.23)}$$

can be rewritten as (see Appendix B)

$$I_1 \approx \mu^{-(n-m+l-2)} e^{-\frac{1}{\mu}} \tag{III.24}$$

since at asymptotically low SNR, we have

$$\mu^p \ll \mu^q \;\; \forall p, q \in \mathbb{N} \text{ such as } p > q. \tag{III.25}$$

So we can say that

$$\text{SNR} \approx \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{l=0}^{2j} C(n,m,i,j,l) \mu^{-(n-m+l-2)} e^{-\frac{1}{\mu}}. \tag{III.26}$$

Using the same considerations as in (III.25), we can restrict this expression to the term with highest value of l which is $l = 2(m-1)$. For this particular value

$$C(n,m,i,j,l) = C(n,m,m-1,m-1,2(m-1)) = \frac{1}{(m-1)!(n-1)!}.$$

As such we end up with

$$\text{SNR} \approx \frac{1}{(m-1)!(n-1)!} \mu^{-(n+m-4)} e^{-\frac{1}{\mu}}. \tag{III.27}$$

This equation can be solved by transforming it into an equation of the type $y = xe^x$ which is solved in terms of the Lambert-W function. More specifically we have three cases:

Case 1: $n + m - 4 < 0$

In this case the solution is given by the main branch of Lambert W function $W_0(.)$ as

$$\frac{1}{\mu} \approx -(n+m-4) W_0 \left(-\frac{(\text{SNR}(m-1)!(n-1)!)^{\frac{1}{n+m-4}}}{n+m-4} \right). \tag{III.28}$$

Case 2: $n + m - 4 = 0$

This is the simplest case where in fact $(m, n) \in \{(2, 2), (1, 3)\}$. In this case μ is simply given by

$$\frac{1}{\mu} \approx -\log(\mathrm{SNR}(n-1)!). \tag{III.29}$$

<u>Case 3</u>: $n+m-4>0$

In this case, the lower branch of the lambert W function is used since the argument given to that function would be negative, and our solution must rationally be converging towards 0 when the power do so. Thus in this case we get

$$\frac{1}{\mu} \approx (n+m-4)W_{-1}\left(-\frac{(\mathrm{SNR}(m-1)!(n-1)!)^{\frac{1}{n+m-4}}}{n+m-4}\right). \tag{III.30}$$

However this solution is meaningless when the argument of $W_{-1}(.)$ function is less than $-\frac{1}{e}$. As such our average power must be beneath a certain value for (III.30) to be valid, and that value is given by

$$\mathrm{SNR} \leq \frac{1}{(m-1)!(n-1)!}\left(\frac{n+m-4}{e}\right)^{(n+m-4)}. \tag{III.31}$$

Using the fact that

$$\lim_{x\to\infty}\frac{W_0(\beta x)}{W_0(x)} = 1 \text{ and } \lim_{x\to 0^-}\frac{W_{-1}(\beta x)}{W_{-1}(x)} = 1 \;\forall\, \beta > 0, \tag{III.32}$$

as shown in [9, Eq. (16),(17)], we can further simplify (III.28) and (III.30). A more simplified solution of (III.27) can be obtained by applying the log function on both sides of (III.27) and recalling that when $\mathrm{SNR} \to 0$ then $\mu \to 0$. We can also neglect $\log((m-1)!(n-1)!)$ besides $\log(\mathrm{SNR})$ at asymptotically low SNR for fixed m and n. In this case, we get

$$\mu \approx -\frac{1}{\log(\mathrm{SNR})}. \tag{III.33}$$

The water level μ being characterized, let us compute the capacity. From (III.8), we have

$$C = m\,\mathbb{E}((\log(\mu\lambda))^+). \tag{III.34}$$

Using again the PDF of the eigenvalues given in (III.11), we obtain

$$C = \sum_{i=0}^{m-1}\sum_{j=0}^{i}\sum_{l=0}^{2j} C(n,m,i,j,l)\int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda))\lambda^{n-m+l}e^{-\lambda}\,d\lambda.$$

Using [8, Eq. (8.350.2) p. 949], we can write

$$\int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda))\lambda^{n-m+l}e^{-\lambda}\,d\lambda = \log(\mu)\Gamma\left(n-m+l+1,\frac{1}{\mu}\right) \tag{III.35}$$

$$+\int_{\frac{1}{\mu}}^{\infty} \log(\lambda)\lambda^{n-m+l}e^{-\lambda}\,d\lambda. \tag{III.36}$$

Letting $t=\mu\lambda$ in the second term of (III.35), we get

$$\int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda))\lambda^{n-m+l}e^{-\lambda}\,d\lambda = \log(\mu)\Gamma\left(n-m+l+1,\frac{1}{\mu}\right)$$

$$+\log\left(\frac{1}{\mu}\right)\Gamma\left(n-m+l+1,\frac{1}{\mu}\right)$$

$$+\left(\frac{1}{\mu}\right)^{n-m+l+1}\int_{\frac{1}{\mu}}^{\infty} \log(t)t^{n-m+l}e^{-\frac{t}{\mu}}\,dt. \tag{III.37}$$

From [10, Eq. (64)], we know that

$$\int_{\frac{1}{\mu}}^{\infty} \log(t)t^{n-m+l}e^{-\frac{t}{\mu}}\,dt = \frac{(n-m+l)!}{\left(\frac{1}{\mu}\right)^{n-m+l+1}}\sum_{k=0}^{n-m+l}\frac{\Gamma(k,\frac{1}{\mu})}{k!} \tag{III.38}$$

In addition to that, using similar techniques as in (III.25) for (III.37), we can write

$$\int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda))\lambda^{n-m+l}e^{-\lambda}\,d\lambda \approx \left(\frac{1}{\mu}\right)^{n-m+l+1} e^{-\frac{1}{\mu}}.$$

Thus, the capacity at low SNR can be written as

$$C \approx \sum_{i=0}^{m-1}\sum_{j=0}^{i}\sum_{l=0}^{2j} C(n,m,i,j,l)\left(\frac{1}{\mu}\right)^{n-m+l+1} e^{-\frac{1}{\mu}}. \tag{III.39}$$

In (III.39) we recognize the expression of the SNR given in (III.26). Thus we can write

$$C \approx \frac{\text{SNR}}{\mu}. \tag{III.40}$$

Combining (III.28), (III.29), (III.30), and (III.32) along with (III.40) gives (III.21), whereas (III.22) follows from (III.33) and (III.40). □

From (III.40), we can obviously see the tremendous impact of CSI-T on MIMO systems at low SNR. Indeed, instead of rSNR, the capacity is now $\frac{1}{\mu}$SNR where μ goes to 0 along with the SNR confirming that at low SNR, the gap between the full CSI capacity and the CSI-R capacity of MIMO Rayleigh channels grows to infinity. But now, we know how its grows to infinity ($\frac{1}{r\mu}$).

III.4 Noisy CSI-T Capacity At Low SNR

Now let us consider the situation when only an estimated version of the channel is available at the transmitter. Many estimation techniques exist among which we have the linear MMSE estimator. In this method, we send a matrix sequence S and we receive Y which is defined as: $Y = HS + N$ where H is our unknown and N is the channel additive Gaussian noise. The MMSE algorithm will try to minimize a certain cost function defined as

$$J_{LS}(H) = (HS - Y)(HS - Y)^{\dagger}.$$

The minimization of this function yields the channel estimator $\hat{H}$ as

$$\hat{H} = (S^{\dagger}S)^{-1}S^{\dagger}Y.$$

In the previous section we saw that availability of CSI at the transmitter only affects the channel capacity at low SNR. So we will now investigate the loss of performance due to error in the estimation of CSI at the transmitter in the low SNR regime. We suppose that the channel can be expressed as

$$H = \hat{H} + \tilde{H} \tag{III.41}$$

where $\tilde{H}$ is the error matrix independent of $\hat{H}$ and whose entries are $CN(0, \alpha)$. So the entries of $\hat{H}$ are $CN(0, 1 - \alpha)$, where CN means circularly symmetric Gaussian complex.

From (III.41), using singular value decomposition, we can deduce that

$$\lambda_{max} \overset{d}{=} \hat{\lambda}_{max} + \tilde{\lambda}_{max} \tag{III.42}$$

where λ_{max}, $\hat{\lambda}_{max}$ and $\tilde{\lambda}_{max}$ are respectively the maximum eigenvalues of $HH^\dagger$, $\hat{H}\hat{H}^\dagger$, $\tilde{H}\tilde{H}^\dagger$; and $X \overset{d}{=} Y$ means X and Y are equal in distribution.

The effect of estimation error on the low SNR capacity of MIMO channels is stated in the following theorem.

Theorem. *The capacity of MIMO channels with Rayleigh fading at low SNR when only a MMSE estimation of the channel is available at the transmitter but the channel is perfectly known at the receiver is as follows*

$$C_0^\alpha \approx \begin{cases} -(1-\alpha)(n+m-4)\ SNR\ W_0\left(((1-\alpha)SNR)^{\frac{1}{n+m-4}}\right) & \text{if } n+m-4<0, \\ -(1-\alpha)\ SNR\ \log((1-\alpha)SNR) & \text{if } n+m-4=0, \\ -(1-\alpha)(n+m-4)\ SNR\ W_{-1}\left(-((1-\alpha)SNR)^{\frac{1}{n+m-4}}\right) & \text{if } n+m-4>0, \end{cases} \tag{III.43}$$

$$\approx -(1-\alpha)\ SNR\ \log((1-\alpha)SNR) \tag{III.44}$$

where α is the estimation error variance, $W_0(.)$ and $W_{-1}(.)$ are the main and lower branches of the Lambert-W function respectively.

Proof. To prove this theorem, we will derive asymptotically identical upper and lower bounds on this capacity.

- Upper bound on C_0^α

  The channel model in previous section becomes

  $$y = (\hat{H} + \tilde{H})x + n \tag{III.45}$$

  where x is the transmitted signal, y the received one and n the Gaussian noise. (III.45) can be rewritten as $y = y_1 + y_2$ where $y_1 = \hat{H}x + \sqrt{\epsilon}\, n_1$ and $y_2 = \tilde{H}x + \sqrt{1-\epsilon}\, n_2$. Here the entries of n_1 and n_2 are $CN(0,1)$ and $\epsilon \in]0,1[$. Since we have $I(x, y_1{+}y_2) \leq I(x, y_1){+}I(x, y_2)$, the capacity of (III.45) is upper-bounded by the sum capacity of two separate channels having outputs y_1 (channel 1) and y_2 (channel 2). The capacity of channel 1 (C_1) is given by Theorem III.3.2 (previous section) since $\hat{H}$ is perfectly known and that of channel 2 (C_2) is $r\frac{\alpha}{\epsilon}SNR$ because $\tilde{H}$ is unknown (see Section II.3.2). From the asymptotic properties of the Lambert function and the log function, C_2 can be neglected besides $C1$. Furthermore, we take the limit when $\epsilon \to 0$ and we get

$$C_0^\alpha \tilde{<} (1-\alpha)C_0^2 \tag{III.46}$$

where C_0^2 is the capacity when the channel is perfectly known.

- Lower bound on C_0^α
  Let us consider an on-off scheme having the following power profile

$$P(\hat{\lambda}_{max}) = \begin{cases} P_0 & \text{if } \hat{\lambda}_{max} \geq \tau \\ 0 & \text{otherwise} \end{cases}$$

where τ is a threshold, $\hat{\lambda}_{max}$ is the maximum eigenvalue of $\hat{H}\hat{H}^\dagger$ and P_0 satisfies the average power constraint

$$P_0 = \frac{\text{SNR}}{1 - F_{\lambda_{max}}(\tau)}$$

The rate of this transmission scheme is given by

$$\begin{aligned} R_\alpha &= \mathop{\mathbf{E}}_{\lambda_{max}, \hat{\lambda}_{max} \geq \tau} [\log(1 + \lambda_{max} P_0)] && \text{(III.47)} \\ &\geq \mathop{\mathbf{E}}_{\lambda_{max} \geq \tau, \hat{\lambda}_{max} \geq \tau} [\log(1 + P_0 \tau)] && \text{(III.48)} \\ &= \mathop{\mathbf{E}}_{\hat{\lambda}_{max} \geq \tau} [\log(1 + P_0 \tau)] && \text{(III.49)} \\ &\approx P_0 \tau \left(1 - F_{\hat{\lambda}_{max}}(\tau)\right) && \text{(III.50)} \\ &= \text{SNR}\, \tau = C_0 \geq (1-\alpha) C_0 && \text{(III.51)} \end{aligned}$$

where (III.49) follows from (III.42) and (III.50) is due to the fact that $P_0\tau \to 0$ when SNR $\to 0$.

Combining (III.46) and (III.51) gives Theorem III.4. □

III.5 Numerical Results

In Fig. III.1, Fig. III.2, Fig. III.3 and Fig. III.4; the exact capacity is computed using standard root-finding algorithms to evaluate the water-filling level μ and then using

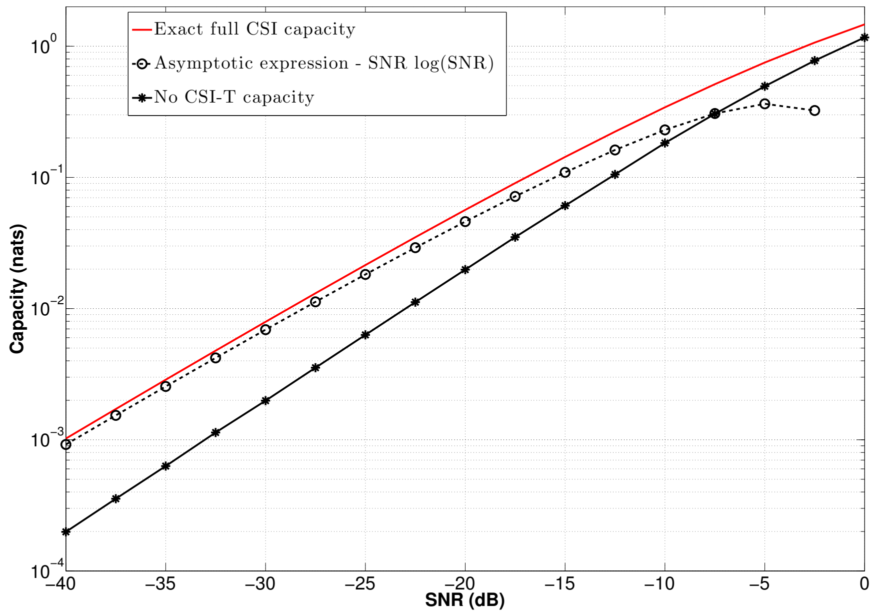

Figure III.1: 2 transmit and 2 receive antennas channel capacity at Low-SNR in nats per channel use (npcu) versus SNR in dB.

numerical integration. The no CSI-T capacity is obtained by Monte-Carlo simulations. The tightness of the expressions given in (III.21) and (III.22) is clearly visible in Fig. III.1, Fig. III.2, and Fig. III.3. The growing gap between the full CSI capacity and the no-CSI capacity when the SNR converges towards zero stress on the fact that CSI at the transmitter affects considerably the capacity at low SNR. Fig. III.1 depicts the particular case of $n = m = 2$ in which the two expressions (III.21) and (III.22) become identical. In Fig. III.2, the first asymptotic expression uses the main branch of the Lambert function since $n + m < 4$; we can also note that in this case the second asymtotic expression approaches the exact capacity from above. But in Fig. III.3, the lower branch is used instead because $n + m > 4$ and the second asymptotic expression approaches the exact capacity from below. Fig. III.4 describes the full CSI capacity for 3-3 MIMO channel and it shows numerically the validity of the high SNR asymptotic expression given in (III.20). The figure III.5 presents the 2×2 MIMO channel capacity with full CSI at the transmitter, then with noisy CSI-T for different estimation error variances, it provides numerical validation to Theorem III.4..

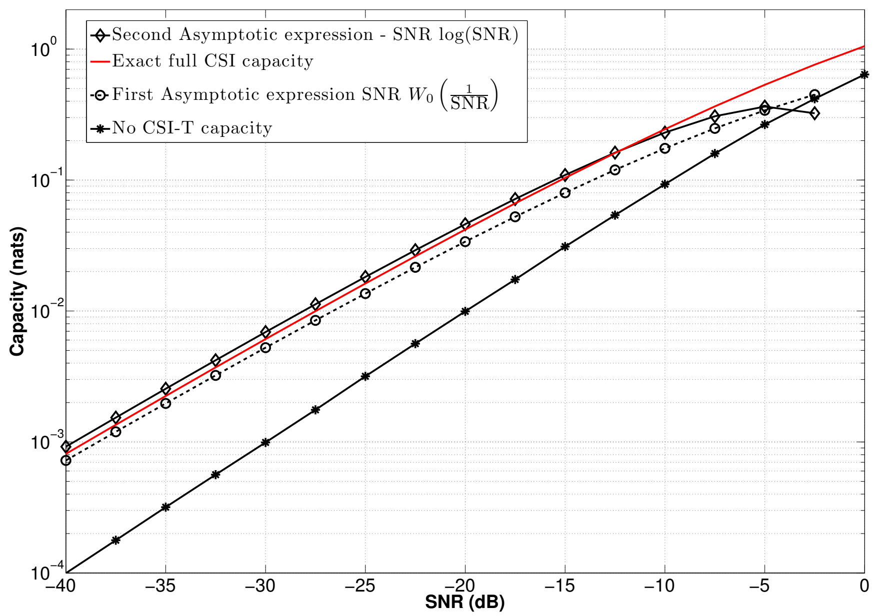

Figure III.2: 2 transmit and 1 receive antennas channel capacity with full CSI at Low-SNR in nats per channel use (npcu) versus SNR in dB.

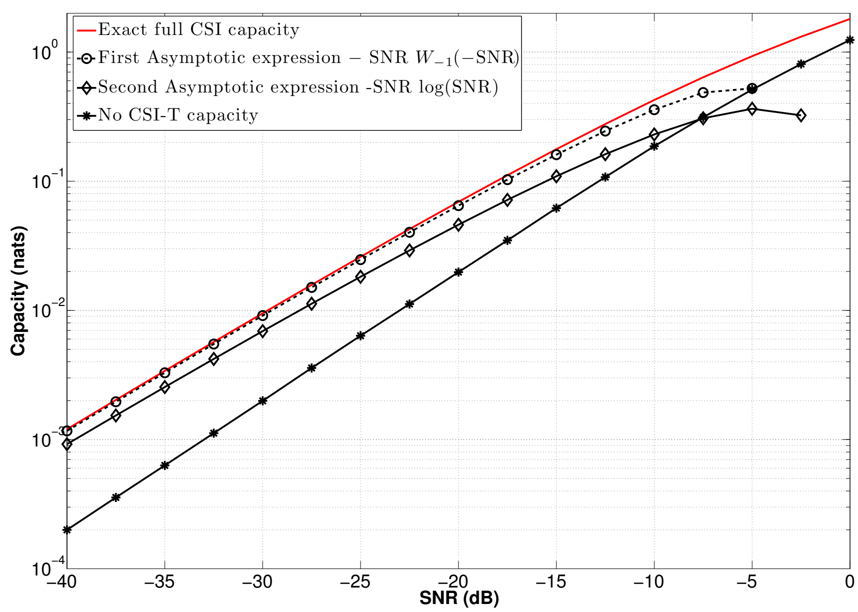

Figure III.3: 3 transmit and 2 receive antennas channel capacity at Low-SNR in nats per channel use (npcu) versus SNR in dB.

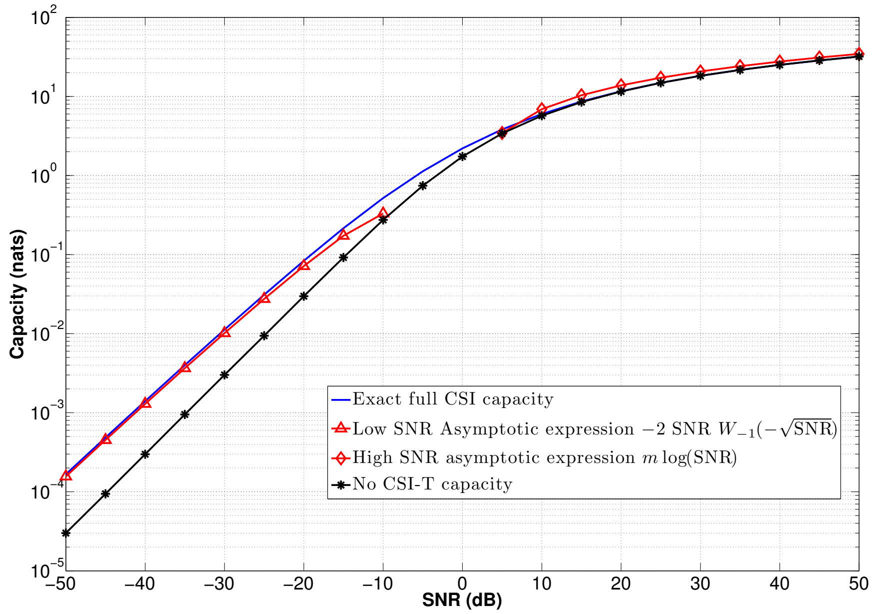

Figure III.4: 3 transmit and 3 receive antennas channel capacity with full CSI in nats per channel use (npcu) versus SNR in dB.

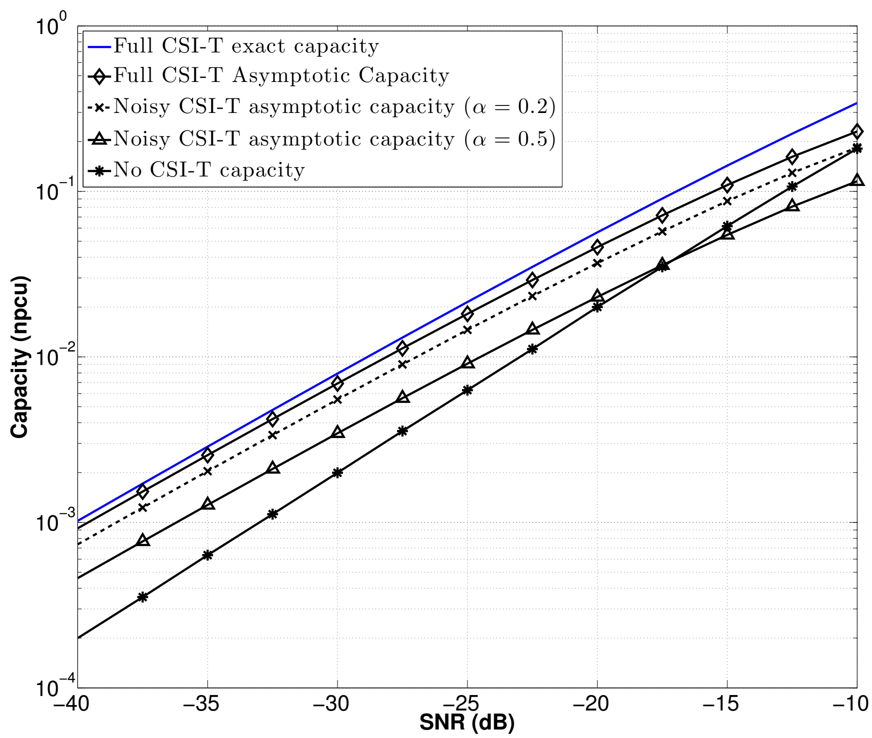

Figure III.5: 2 transmit and 2 receive antennas channel capacity with noisy CSI-T at Low-SNR in nats per channel use (npcu) versus SNR in dB.

III.6 Conclusion

The MIMO channel under Rayleigh fading has been investigated in this chapter with perfect CSI and then noisy CSI at the transmitter and with perfect CSI at the receiver. The general expression of the capacity for any SNR has been given and then asymptotic analysis has been to provide useful insights on MIMO channels capacity. The results provided for the low SNR regime constitute an important part of the graduation project outputs. It has been demonstrated that availability of CSI at the transmitter has an impact only in the low SNR regime. Also, the multiple antennas effect has been shown to decrease as the SNR goes to 0. In other terms, at asymptotically low SNR, MIMO and SISO sytems have the same capacity. This result has led us to extend our study to SISO channels undergoing Log-normal shadowing in order to obtain simple low SNR expressions, the results are presented in the next chapter.

Chapter IV

Low SNR Capacity of a channel undergoing Log-normal shadowing

IV.1 Introduction

The impact of CSI on channel capacity has been thoroughly studied for Rayleigh, Rician and Nakagami fading which had not been the case for log-normal fading until recently. However, this model describes the best environments undergoing slow fading such as shadowing in urban areas [11] or small-scale fading in ultra-wideband (UWB) communications [12].

In [13] upper and lower bounds to the log-normal SISO channel capacity are provided. In [14] closed-form approximations of that capacity are given but considering no channel knowledge at the transmitter. The adaptive transmission is considered in [15] and closed-form approximations are provided especially for optimal power adaptation but those expressions do not allow any insight on how evolve this capacity either at high or low signal-to-noise ratio (SNR).

In this paper we further investigate the ergodic capacity of Log-normal fading channels with full channel state information both at the transmitter and the receiver at asymptotically low SNR. That capacity is shown to scale essentially as $\lambda\,\mathrm{SNR}$ where λ is the water-filling level satisfying the power constraint. An asymptotically closed-form expression of λ is also provided. We also provide an on-off transmission which is capacity achieving at asymptotically low SNR.

IV.2 General Expression

We plan to investigate the low SNR capacity of a log-normal shadowing channel when the channel is perfectly known to both the transmitter and the receiver. The channel under consideration here can be represented as follows

$$y = hx + n \tag{IV.1}$$

when $y \in \mathbb{C}$ is the channel output, $x \in \mathbb{C}$ its input, n the gaussian noise and h representing the shadowing coefficient. As such, the probability density function (PDF) of h is given in [11] as

$$f_h(t) = \frac{\xi}{\sqrt{2\pi}\sigma t} \exp\left(-\frac{(\xi \log(t) - \mu)^2}{2\sigma^2}\right) \tag{IV.2}$$

where μ and σ are respectively the mean and the variance of h and $\xi = \frac{10}{\log(10)}$.

Since h is perfectly known at both sides of the link, the expression of the capacity is given by

$$C = \mathop{\mathbf{E}}_{h^2}\left[\log\left(\frac{t}{\lambda}\right)^+\right] \tag{IV.3}$$

where λ is the water-filling level chosen to meet the power constraint

$$P_{avg} = \mathop{\mathbf{E}}_{h^2}\left[\left(\frac{1}{\lambda} - \frac{1}{t}\right)^+\right] \tag{IV.4}$$

IV.3 Low SNR Expression

The expression in (IV.3) can be evaluated numerically but it does not provide a clear insight of the impact of SNR on the capacity. We next plan to characterize this capacity at low SNR by using asymptotic analysis. The results are stated in the following theorem.

Theorem. *The capacity of a wireless channel undergoing log-normal shadowing at asymptotically low SNR when both the transmitter and the receiver know perfectly the channel is as follows*

$$C_{ln} \approx e^{\frac{\sigma}{\xi}\sqrt{-\log(SNR^2)}}\ SNR \tag{IV.5}$$

where $\xi = \frac{10}{\log(10)}$, σ^2 *is the shadowing coefficient standard deviation and* μ *its mean.*

Proof. First let us characterize the water-filling level λ in (IV.4). Using (IV.2) in (IV.4), we have

$$\text{SNR} = \int_{\lambda}^{\infty} \left(\frac{1}{\lambda} - \frac{1}{t}\right) \left[\frac{\xi}{\sigma t \sqrt{2\pi}} \exp\left(-\frac{(\xi \log(t) - \mu)^2}{2\sigma^2}\right)\right] dt \tag{IV.6}$$

If we note $x = \frac{\xi \log(t) - \mu}{\sigma}$ and $\tau = \frac{\xi \log(\lambda) - \mu}{\sigma\sqrt{2}}$, we get $t = e^{\left(\frac{1}{\xi}(\sigma x + \mu)\right)}$ and $\frac{dt}{\sigma t} = \frac{dx}{\xi}$. So (IV.6) becomes

$$\begin{aligned}\text{SNR} &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\lambda} \int_{\sqrt{2}\tau}^{\infty} e^{-\frac{x^2}{2}} dx - \int_{\sqrt{2}\tau}^{\infty} e^{-\left(\frac{1}{\xi}(\sigma x + \mu)\right)} e^{-\frac{x^2}{2}} dx\right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\lambda}\sqrt{\frac{\pi}{2}}\,\mathbf{erfc}(\tau) - e^{a^2 - b}\sqrt{\frac{\pi}{2}}\,\mathbf{erfc}(a + \tau)\right]\end{aligned} \tag{IV.7}$$

where $a = \frac{\sigma}{\xi\sqrt{2}}$, $b = \frac{\mu}{\xi}$, and $\mathbf{erfc}(.)$ is the complementary error function defined in [8, Eq. (8.250.4)]. Having from [16] that $\lambda \to +\infty$ at low SNR and using the asymptotic expansion of the complementary error function given in [8, Eq. (8.254)], we get the following asymptotic expression

$$\begin{aligned}\text{SNR} &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} \left[\frac{1}{\lambda} e^{-\tau^2} \left(\frac{1}{\sqrt{\pi}\tau} - \frac{1}{2\sqrt{\pi}\tau^3} + o\left(\frac{1}{\tau}\right)^4\right)\right. \\ &\quad \left. - e^{a^2 - b} e^{-(a+\tau)^2} \left(\frac{1}{\sqrt{\pi}\tau} - \frac{a}{\sqrt{\pi}\tau^2} + o\left(\frac{1}{\tau}\right)^2\right)\right] \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}} \left[\frac{1}{\lambda} e^{-\tau^2} \left(\frac{1}{\sqrt{\pi}\tau} - \frac{1}{2\sqrt{\pi}\tau^3} + o\left(\frac{1}{\tau}\right)^4\right)\right. \\ &\quad \left. - e^{(-\tau^2 - \log(\lambda))} \left(\frac{1}{\sqrt{\pi}\tau} - \frac{a}{\sqrt{\pi}\tau^2} + o\left(\frac{1}{\tau}\right)^2\right)\right] \\ &= \frac{1}{\sqrt{2\pi}} \frac{a}{\sqrt{2}} \frac{1}{\lambda} e^{-\tau^2} \left[\frac{1}{\tau^2} + o\left(\frac{1}{\tau}\right)^2\right]\end{aligned} \tag{IV.8}$$

which yields us the following asymptotic equation

$$\text{SNR} \approx \frac{1}{\sqrt{2\pi}} \frac{\sigma}{2\xi} \frac{1}{\lambda} \frac{e^{-\left(\frac{\xi \log(\lambda) - \mu}{\sigma\sqrt{2}}\right)^2}}{\left(\frac{\xi \log(\lambda) - \mu}{\sigma\sqrt{2}}\right)^2} \tag{IV.9}$$

Applying the log function to both sides of (IV.8), we get

$$\log(\mathrm{SNR}) \approx \log\left[\frac{1}{\sqrt{2\pi}}\frac{\sigma}{2\xi}\right] - \log(\lambda) - \log\left(\left(\frac{\xi\log(\lambda)-\mu}{\sigma\sqrt{2}}\right)^2\right) \tag{IV.10}$$

$$-\frac{\xi}{2\sigma^2}(\log(\lambda))^2 + \frac{\mu\xi}{\sigma}\log(\lambda) - \frac{\mu^2}{2\sigma} \tag{IV.11}$$

$$\approx -\frac{\xi}{2\sigma^2}(\log(\lambda))^2 + \left(\frac{\mu\xi}{\sigma}-1\right)\log(\lambda) \tag{IV.12}$$

Solving (IV.12) as a quadratic equation, we deduce that

$$\lambda \approx e^{\frac{\sigma^2}{\xi^2}\left(\sqrt{\left(1-\frac{\mu\xi}{\sigma}\right)^2 - 2\frac{\xi^2}{\sigma^2}\log(\mathrm{SNR})} - \left(1-\frac{\mu\xi}{\sigma}\right)\right)} \tag{IV.13}$$

$$\approx e^{\frac{\sigma}{\xi}\left(-\left(\frac{\sigma}{\xi}-\mu\right)+\sqrt{\left(\frac{\sigma}{\xi}-\mu\right)^2-2\log(\mathrm{SNR})}\right)} \tag{IV.14}$$

which can be simplified further into

$$\lambda = e^{\frac{\sigma}{\xi}\sqrt{-\log(\mathrm{SNR}^2)}} \tag{IV.15}$$

Now let us evaluate the capacity. From (IV.3) and (IV.2), we have

$$C = \int_\lambda^\infty \log\left(\frac{t}{\lambda}\right)\left[\frac{\xi}{\sigma t\sqrt{2\pi}}\exp\left(-\frac{(\xi\log(t)-\mu)^2}{2\sigma^2}\right)\right]dt \tag{IV.16}$$

If we note $x = \frac{\xi\log(t)-\mu}{\sigma}$ and $\tau = \frac{\xi\log(\lambda)-\mu}{\sigma\sqrt{2}}$, we get

$$C = \frac{1}{\sqrt{2\pi}}\left[-\log(\lambda)\int_{\sqrt{2}\tau}^\infty e^{-\frac{x^2}{2}}dx + \frac{1}{\xi}\int_{\sqrt{2}\tau}^\infty(\sigma x+\mu)\,e^{-\frac{x^2}{2}}dx\right] \tag{IV.17}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\mu}{\xi}-\log(\lambda)\right)\sqrt{\frac{\pi}{2}}\,\mathbf{erfc}(\tau) + \frac{\sigma}{\xi}e^{-\tau^2}\right] \tag{IV.18}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{e^{-\tau^2}}{\tau}\left[\left(\frac{\mu}{\xi}-\log(\lambda)\right)\frac{1}{\sqrt{2}}\left(1-\frac{1}{2\tau^2}+o\left(\frac{1}{\tau}\right)^4\right) - \frac{\sigma}{\xi}\left(\frac{\xi\log(\lambda)-\mu}{\sigma\sqrt{2}}\right)\right]$$

$$\approx \frac{1}{\sqrt{2\pi}}\frac{e^{-\tau^2}}{\tau}\left[-\frac{1}{2\sqrt{2}\tau^2}\left(\frac{\mu}{\xi}-\log(\lambda)\right)\right]$$

$$\approx \frac{1}{\sqrt{2\pi}}\frac{e^{-\tau^2}}{\tau^2}\left[-\frac{1}{2\sqrt{2}}\frac{\frac{\mu}{\xi}-\log(\lambda)}{\frac{\xi\log(\lambda)-\mu}{\sigma\sqrt{2}}}\right] \tag{IV.19}$$

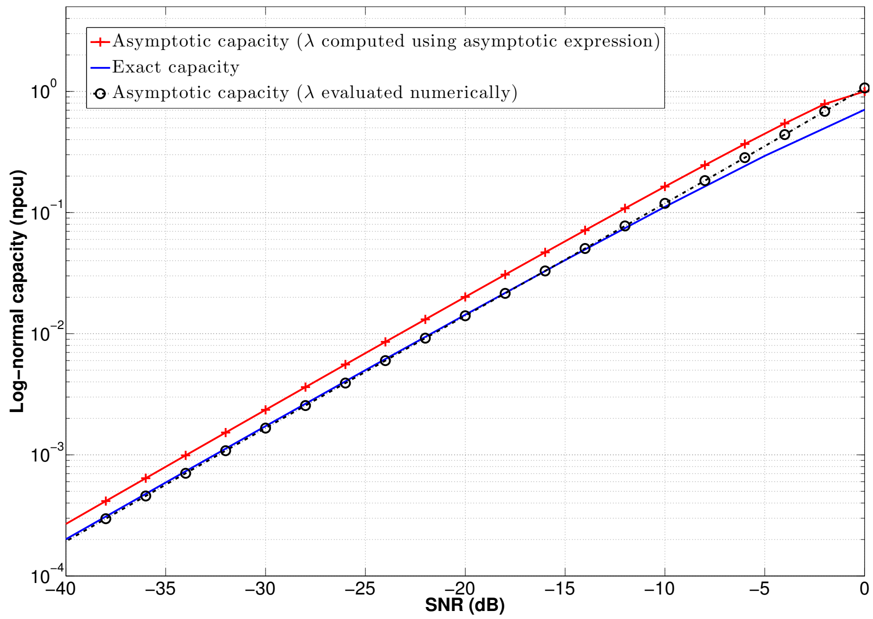

Figure IV.1: Log-normal channel capacity at Low-SNR in nats per channel use (npcu) versus SNR in dB.

By simplifying, we end up with

$$C \approx \frac{1}{\sqrt{2\pi}} \frac{\sigma}{2\xi} \frac{e^{-\left(\frac{\xi \log(\lambda)-\mu}{\sigma\sqrt{2}}\right)^2}}{\left(\frac{\xi \log(\lambda)-\mu}{\sigma\sqrt{2}}\right)^2} \approx \lambda \, \text{SNR} \tag{IV.20}$$

□

IV.4 Numerical Results

Figure IV.1 presents the low SNR capacity for a SISO channel undergoing log-normal shadowing with parameters $\mu = 0$ and $\sigma = 1$. We can see that when λ is computed by numerically solving (IV.9), the asymptotic capacity overlaps almost always the exact capacity. Even when λ is computed using the asymptotic expression (IV.15), the asymptotic capacity is still tight to the exact one.

IV.5 Conclusion

In this chapter, taking advantage of the results obtained for the Rayleigh fading MIMO channel, we have investigated the SISO channel undergoing Lognormal shadowing. We characterize its capacity at low SNR and derived closed-form asymptotic expressions to that capacity. We saw that the capacity scales essentially as λ SNR where λ goes to $+\infty$ when SNR $\to 0$. Now, in the next chapter, we will exploit all these results and construct asymptotically capacity-achieving low-SNR transmission schemes.

Chapter V

On-Off transmission scheme

V.1 Introduction

Many systems nowadays operate at low SNR taking advantage of large available bandwidth as in wideband communications [17] or because of their own nature and their intrinsic purposes like in sensor networks. Due to this fact, studying systems behavior in the low SNR regime has become inevitable. Moreover, new schemes and techniques must be found to provide the best performances in this regime. A now common result in wireless communications is that the on-off scheme appears to be optimal in the low SNR regime for many systems. This result is rational because we dispose of low power at the transmitter so one should remain silent and send opportunistically when the channel is very good using a Gaussian codebook and a fixed transmit power. In clear, the on-off scheme uses an on-off power control defined as follows:

$$P(h) = \begin{cases} P_0 & \text{if } h \geq \tau \\ 0 & \text{otherwise} \end{cases}$$

where h is the channel gain, τ a certain threshold to be defined and P_0 a constant transmit power chosen to meet the average power constraint. So in order to achieve capacity, τ and P_0 must be chosen with care. However, this choice is often difficult as the analytic expression of the capacity does not give sufficient insights on the impact of different parameters. Since we are at low SNR, we will use the framework of our asymptotic analysis done in the previous chapters to design on-off schemes for different configurations.

V.2 Perfect CSI-T Case In MIMO Rayleigh Fading Channel

In this section, we will focus on the full CSI case. The results in III.3.2 have demonstrated us that the capacity becomes more unaffected by multiple antennas as SNR goes to 0. So this is equivalent in saying that the capacity is close to that of a SISO channel. Using this insight, one can think of transmitting along the maximum eigenvalue of the Wishart matrix $HH^\dagger$ only in order to get the best of the channel. More specifically, the on-off power control scheme is

$$P(\lambda_{max}) = \begin{cases} P_0 & \text{if } \lambda_{max} \geq \tau \\ 0 & \text{otherwise} \end{cases} \tag{V.1}$$

where P_0 satisfies the average power constraint

$$P_0 = \frac{\text{SNR}}{1 - F_{\lambda_{max}}(\tau)}$$

The cumulative distribution function of the largest eigenvalue of a Wishart matrix is given in [18, eq. (6)] by

$$F_{\lambda_{max}}(x) = \frac{det(\Psi(x))}{\prod_{k=1}^{m} \Gamma(n-k+1)\Gamma(m-k+1)}$$

where $\Psi(x)$ is a $m \times m$ matrix defined as follows:

$$\Psi_{(i,j)}(x) = \gamma(n-m+i+j-1, x); i,j = 1, ..., m$$

where $\gamma(.)$ is the lower incomplete gamma function [8, Eq. (8.350.1) p. 949]. Taking advantage of our framework in III.3.2, we choose τ to be the inverse of the water-filling level μ which is the solution of (III.27), its low-SNR expression is given by (III.28) , (III.29) or (III.30) depending on the values of m and n. Therefore, the achievable rate, say R, of this on-off scheme is given by

$$\begin{aligned} R &= \mathbb{E}_{\lambda_{max}} \left(\log(1 + P(\lambda_{max})\lambda_{max})\right) \\ &= \int_{\tau}^{\infty} \log(1 + P_0\lambda) p_{\lambda_{max}}(\lambda) d\lambda \qquad (V.2) \\ &\geq \log(1 + P_0\tau) \int_{\tau}^{\infty} p_{\lambda_{max}}(\lambda) d\lambda \qquad (V.3) \end{aligned}$$

Now let us note that $F_{\lambda_{max}}(\tau) \leq F_{\lambda}(\tau)$. Thus

$$P_0\tau = \frac{\mathrm{SNR}\tau}{1 - F_{\lambda_{max}}(\tau)} \leq \frac{\mathrm{SNR}\tau}{1 - F_{\lambda}(\tau)}.$$

From (III.27) and considering the value of τ we have chosen, it can be easily verified that $\mathrm{SNR}\tau \approx K_1\tau^{n+m-3}e^{-\tau}$. Also using the PDF of λ in (II.6) we can show that $1 - F_{\lambda}(\tau) \approx K_2\tau^{n+m-2}e^{-\tau}$ for $\tau \to \infty$. K_1 and K_2 are positive real constants. So, it becomes clear that

$$\lim_{\tau\to\infty} P_0\tau = 0 \qquad (V.4)$$

Then combining (V.3) and (V.4), and recalling that $\log(1 + x) \approx x$; the proposed on-off achievable rate can be lower-bounded by

$$R \geq P_0\tau(1 - F_{\lambda_{max}}(\tau)) = \mathrm{SNR}\ \tau \qquad (V.5)$$

The right hand side (RHS) of (V.5) is nothing but the asymptotic capacity expression given by (III.40) which ensures that the proposed on-off scheme is asymptotically capacity-achieving. In fact this on-off scheme is still capacity achieving if we choose any of the eigenvalues (different distribution than that of the maximum eigenvalue) at each realization of the channel and the proof is similar. This means that we can use only one antenna at both sides of the channel and still achieve this capacity. So for this on-off scheme, we only need one bit feedback from the receiver stating whether we transmit or not.

V.3 Full CSI Case In SISO Channel With Log-normal Shadowing

Another interesting case we studied during this internship was the Log-normal shadowing. In the previous chapter, we derived low SNR asymptotic characterization of the capacity under such channel. We now construct an on-off scheme that we prove asymptotically capacity achieving. The idea is is the same as before; we remain silent except when the channel is better than a certain threshold. This is achieved by the following power control

$$P(h) = \begin{cases} P_0 & \text{if } h \geq \lambda \\ 0 & \text{otherwise} \end{cases}$$

where h is the Log-normal channel gain whose distribution is given in (IV.2), λ is the water-filling level solution of equation (IV.9) and P_0 satisfies the average power constraint

$$P_0 = \frac{\text{SNR}}{1 - F_h(\lambda)}$$

The cumulative distribution function (CDF) of the log-normal coefficient is given by

$$F_h(t) = \frac{1}{2}\,\mathbf{erfc}\left(\frac{\xi \log(t) - \mu}{\sigma\sqrt{2}}\right) \tag{V.6}$$

Let us evaluate the achievable rate R of this scheme. We first have

$$\begin{aligned} R &= E_h\left[\log(1 + P(h)h)\right] && \text{(V.7)} \\ &= \int_{\lambda}^{\infty} \log(1 + P_0 t) f_h(t)dt && \text{(V.8)} \\ &\geq \log(1 + P_0\lambda)\left(1 - F_h(\lambda)\right) && \text{(V.9)} \end{aligned}$$

From (IV.7) we have

$$\lambda\,\text{SNR} = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\lambda}\sqrt{\frac{\pi}{2}}\,\mathbf{erfc}(\tau) - e^{a^2-b}\sqrt{\frac{\pi}{2}}\,\mathbf{erfc}(a+\tau)\right] \tag{V.10}$$

and we also have

$$1 - F_h(\lambda) = \frac{1}{2}\,\mathbf{erfc}(\tau) \tag{V.11}$$

So

$$P_0\lambda = 1 - \lambda e^{a^2-b}\frac{\mathbf{erfc}(a+\tau)}{\mathbf{erfc}(\tau)} \tag{V.12}$$

Expanding the complementary error function for large values of λ hence for large values of τ we end up with

$$P_0\lambda = 1 - \frac{\left(\frac{1}{\tau} - \frac{a}{\tau^2} + o\left(\frac{1}{\tau}\right)^4\right)}{\left(\frac{1}{\tau} - \frac{1}{2\tau^3} + o\left(\frac{1}{\tau}\right)^4\right)} \tag{V.13}$$

Now we can easily see that $\lim_{\lambda\to\infty} P_0\lambda = 0$ which implies that $\log(1+P_0\lambda) \approx P_0\lambda$. So (V.9) becomes

$$R \geq P_0\lambda\,(1 - F_h(\lambda)) \approx \lambda\ \mathrm{SNR} \tag{V.14}$$

Note that the RHS of (V.14) is the asymptotic expression of the Log-normal SISO channel derived in (IV.20) which proves that this on-off scheme is asymptotically capacity achieving.

V.4 Numerical Results

Figures V.1, V.2 and V.3 present exact capacity along with on-off capacity for different channel configurations. The exact capacity is computed using numerical optimisation algorithms. Fig. V.1 and Fig. V.2 present the Rayleigh fading MIMO channel case for 2 different antennas configurations (2-1 for Fig. V.1 and 3-2 for Fig. V.2). We can clearly see that the rate of the on-off scheme proposed in (V.1) is almost always superposed on the exact capacity, we can conclude that the transmission scheme proposed is really interesting from a practical point of view. Fig. V.3 present the Lognormal SISO channel capacity. The on-off rate has been computed using (V.8). We can see that it approaches the exact capacity from below and is very tight to it. This, again, illustrates how efficient this transmission scheme can be in practical implementation.

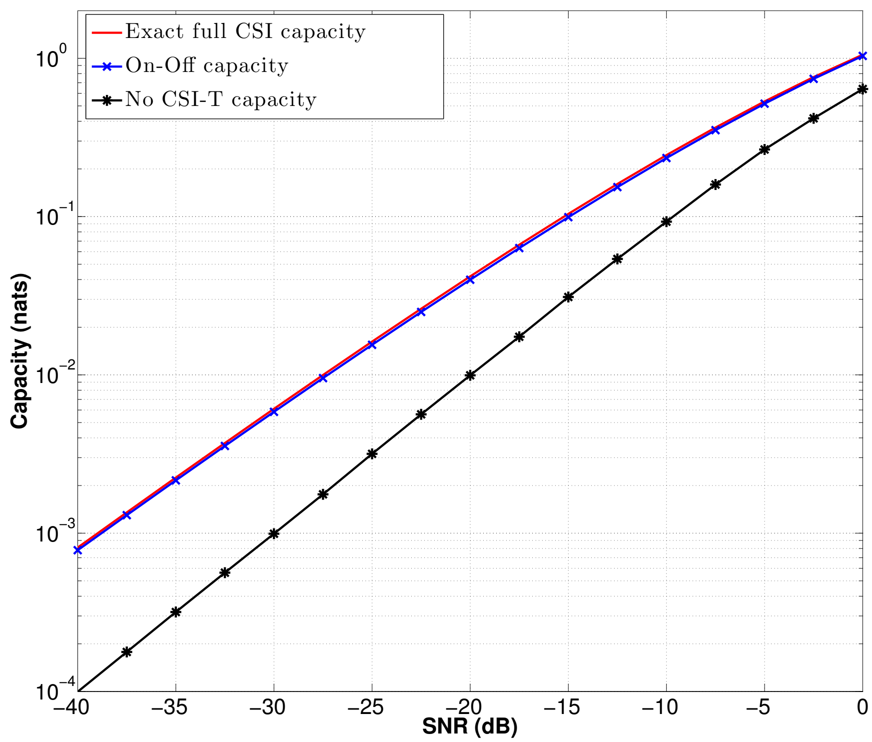

Figure V.1: (2-1) MIMO channel capacity with full CSI and On-Off achievable rate in nats per channel use (npcu) versus SNR in dB.

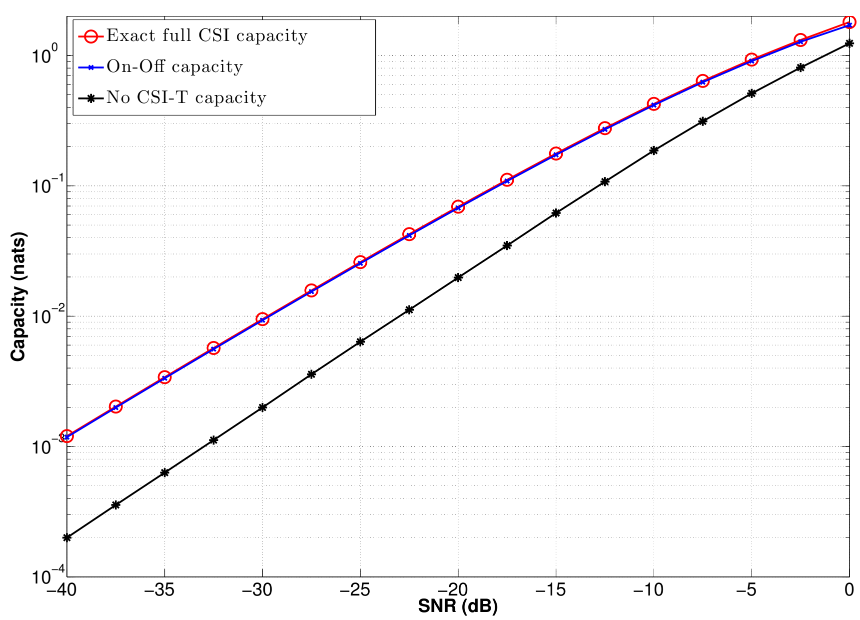

Figure V.2: (3-2) MIMO channel capacity with full CSI and On-Off achievable rate in nats per channel use (npcu) versus SNR in dB.

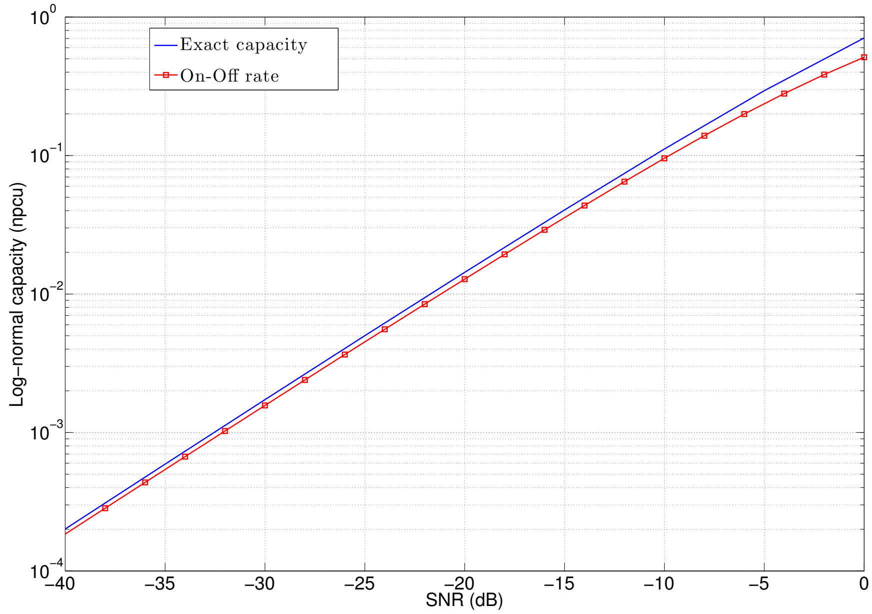

Figure V.3: Lognormal channel capacity with full CSI and On-Off achievable rate for $\mu = 0$dB, $\sigma^2 = 1$ in nats per channel use (npcu) versus SNR in dB.

V.5 Conclusion

We have seen in this chapter that on-off transmission scheme is optimal at asymptotically low SNR for different configurations. We have first shown it for MIMO channels undergoing Rayleigh fading when the channel is perfectly known both at the transmitter and the receiver, then for the SISO channel undergoing Log-normal shadowing. The numerical results presented stress on the validity of the proofs and reveal that the schemes proposed are really interesting from a practical point of view since they are still optimal even for not that low SNR values.

Conclusion

This report presents the research work performed during my graduation Internship at KAUST. The capacities of MIMO Rayleigh fading channel and Lognormal SISO channel were investigated under various assumptions on the availability of channel state information at the transmitter.

The case when the transmitter has no knowledge of the channel but the receiver knows perfectly the channel has been studied in chapter 2. The results obtained especially in the low and high SNR asymptotic analysis demonstrate the interest of using multiple antennas in terms of capacity gain. Then we investigated in chapter 3 the case where both the transmitter and the receiver know perfectly the channel. The capacity has been proven to drastically increase besides the capacity of the no CSIT channel especially at low SNR. We also derived in that chapter interesting asymptotic expressions of the MIMO capacity under Rayleigh fading providing valuable insights on the behavior of that capacity at low SNR. Among those insights, we proved that the effect of multiple antennas vanishes as the SNR goes to 0. Building on these results, we then focused in chapter 4 on the SISO channel undergoing Log-normal shadowing. Accurate asymptotic expressions were derived providing also useful insights. In the end the capacity in the full CSI was shown to scale as λSNR where λ is the Lagrange multiplier appearing when maximizing the mutual information under average power constraint. The insights provided by the results in chapters 3 and 4 enabled us to construct on-off transmission schemes that we proved asymtotically capacity-achieving in chapter 5. Furthermore, the numerical results obtained in chapter 5 demonstrate the usefulness of the proposed schemes even for not that low SNR values from a practical point of view.

This graduation project allowed me to have a great first experience in engineering research. I was able to get in deeper touch with the MIMO technology and understand why it is so praised.

Many extensions can be found to this work among which is considering different fading for the MIMO case (Ricean, Lognormal). Also the ergodic case can be replaced by the notion of coherence time and consequences on the capacity can be derived. We can also investigate other performance indicators than the capacity such as the bit error rate, the outage probability, etc.

APPENDICES

Appendix A

Expectation of the i.i.d Eigenvalue λ of $HH^\dagger$

There are two methods to compute this expectation, either by using the PDF of λ or using the structure of the matrix H itself.

- **First Method**: From the orthogonality relation on Laguerre polynomials [8, Eq. (8.980)], we have

$$\int_0^\infty \left[L_k^{n-m}(\lambda)\right] \lambda^{n-m+1} e^{-\lambda} d\lambda = \frac{(k+n-m)!}{k!}(2k+n-m+1) \tag{A.1}$$

  Using (III.10) and (A.1), we can write

$$\begin{aligned}
\mathbf{E}_\lambda[\lambda] &= \frac{1}{m}\sum_{i=0}^{m-1} \frac{k!}{(n-m+k)!}\frac{(k+n-m)!}{k!}(2k+n-m+1) && \text{(A.2)}\\
&= \frac{1}{m}\sum_{i=0}^{m-1}(2k+n-m+1) && \text{(A.3)}\\
&= \frac{1}{m}\Big(m(n+m-1)+2\frac{m}{2}(m-1)\Big) && \text{(A.4)}\\
&= n && \text{(A.5)}
\end{aligned}$$

- **Second Method**: Let us first consider the case when $r < t$. Since $HH^\dagger$ is a square hermitian matrix, we can write $HH^\dagger = UDU^\dagger$ where U is unitary and D is diagonal and its elements are i.i.d. So we have

$$\mathbf{E}[HH^\dagger] = \mathbf{E}[UDU^\dagger] = U\mathbb{E}[D]U^\dagger = \mathbb{E}[\lambda]I_r \tag{A.6}$$

On the other hand, if we note $h(i,:)$ the i^{th} line of H, we would have

$$\mathbb{E}[h(i,:)h(i,:)*] = \sum_{k=1}^{t} \mathbb{E}[|H_{i,k}|^2] = t \tag{A.7}$$

because H models a Rayleigh fading channel so $\mathbb{E}[|H_{i,j}|^2] = 1\ \forall i,j$. Since the i^{th} diagonal element of $\mathbb{E}[HH^\dagger]$ is $\mathbb{E}[h(i,:)h(i,:)*]$, we get that

$$\mathbb{E}[\lambda] = t \tag{A.8}$$

In the case when $r \geq t$, we consider the matrix $H^\dagger H$ and retrieve a similar result

$$\mathbb{E}[\lambda] = r \tag{A.9}$$

Equations (A.8) and (A.9) can be summarized in

$$\mathbb{E}[\lambda] = n \tag{A.10}$$

Appendix B

Computation of Integral I_1

We first recall the integral being computed here.

$$I_1 = \int_{\frac{1}{\mu}}^{\infty} \left(\mu - \frac{1}{\lambda}\right) \lambda^{n-m+l} e^{-\lambda}\, d\lambda = \mu\Gamma\left(n-m+l+1, \frac{1}{\mu}\right) - \Gamma\left(n-m+l, \frac{1}{\mu}\right).$$

$\Gamma(.,.)$ can be rewritten using [8, eq. (8.352.2) p. 949] as $\Gamma(k,\alpha) = (k-1)! e^{-\alpha} \sum_{j=0}^{k-1} \frac{\alpha^j}{j!}$. Using this expression of the Gamma function, we can write

$$\begin{aligned}
I_1 &= e^{-\frac{1}{\mu}} \left[\mu(n-m+l)! \sum_{j=0}^{n-m+l} \frac{1}{\mu^j j!} - (n-m+l-1)! \sum_{j=0}^{n-m+l-1} \frac{1}{\mu^j j!}\right] \\
&= e^{-\frac{1}{\mu}} \left[\sum_{j=0}^{n-m+l} \frac{(n-m+l)!}{\mu^{j-1} j!} - \sum_{j=0}^{n-m+l-1} \frac{(n-m+l-1)!}{\mu^j j!}\right] \\
&= e^{-\frac{1}{\mu}} \left[(n-m+l)!\mu + \sum_{k=0}^{n-m+l-1} \frac{(n-m+l)!}{\mu^k (k+1)!} - \sum_{j=0}^{n-m+l-1} \frac{(n-m+l-1)!}{\mu^j j!}\right] \\
&= e^{-\frac{1}{\mu}} \left[\frac{1}{\mu^{n-m+l-2}} + \sum_{k=0}^{n-m+l-3} \left(\frac{(n-m+l)!}{\mu^k (k+1)!} - \frac{(n-m+l-1)!}{\mu^k k!}\right) + (n-m+l)!\mu\right]
\end{aligned}$$

Using (III.25) in the last expression of I_1, we obtain

$$I_1 = \int_{\frac{1}{\mu}}^{\infty} \left(\mu - \frac{1}{\lambda}\right) \lambda^{n-m+l} e^{-\lambda}\, d\lambda \approx \mu^{-(n-m+l-2)} e^{-\frac{1}{\mu}}$$

from which we deduce (III.24). Note that in the above derivation, the term $\mu^{-(n-m+l-1)}$ has been canceled out. We take this opportunity to also provide the proof for the special

cases when $n-m+l=0$ and $n-m+l=1$.
For $n-m+l=0$; we have

$$\begin{aligned} I_1 &= \int_{\frac{1}{\mu}}^{\infty} \left(\mu - \frac{1}{\lambda}\right) e^{-\lambda}\, d\lambda \\ &= \mu e^{-\frac{1}{\mu}} - \mathrm{E}_1\left(\frac{1}{\mu}\right) \end{aligned}$$

where $\mathrm{E}_1(.)$ is the exponential integral defined in [19, p. 228]. With the help of [8, eq. (8.215)], we can write

$$\begin{aligned} I_1 &\approx \mu e^{-\frac{1}{\mu}} - \mu e^{-\frac{1}{\mu}} \sum_{j=0}^{J} (-1)^j \mu^j j!, \text{ where } J \in \mathbb{N}^* \\ &\approx \mu^2 e^{-\frac{1}{\mu}} \end{aligned}$$

which corresponds to the previous result.
For $n-m+l=1$; we have

$$\begin{aligned} I_1 &= \int_{\frac{1}{\mu}}^{\infty} \left(\mu - \frac{1}{\lambda}\right) \lambda e^{-\lambda}\, d\lambda = \mu \Gamma\left(2, \frac{1}{\mu}\right) - e^{-\frac{1}{\mu}} \\ &= \mu e^{-\frac{1}{\mu}} \left(1 + \frac{1}{\mu}\right) - e^{-\frac{1}{\mu}} \approx \mu e^{-\frac{1}{\mu}} \end{aligned}$$

which also corresponds to the previous results. Note that we used again [8, eq. (8.352.2) p. 949] here.

Appendix C

Papers Submitted and Under Preparation

• Abdoulaye Tall, Zouheir Rezki and Mohamed-Slim Alouini, "MIMO Channel Capacity with Full CSI at Low SNR", *Accepted for publication in IEEE Wireless Communication Letters*, Jun. 2012.

• Abdoulaye Tall, Zouheir Rezki and Mohamed-Slim Alouini, "Log-normal channel capacity characterization at low SNR with full CSI", *under preparation*.

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