Tunisia Polytechnic School

جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
Science and Technology

Option: **S**ignals and **S**ystems (**SISY**)

Graduation Project Defense

Low SNR Characterization of the Capacity of Generalized fading Channels

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Vis-à-Vis: Dr. Ines BOUSNINA

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Introduction

Figure : Ultra Wideband communications

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Figure : Sensor Network

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Figure : Sensor Network

⇒ **Need to characterize wireless communications systems at low SNR**

Outline

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MIMO Channel with Rayleigh Fading

MIMO Rayleigh Channel Model

$$y = Hx + n$$

where

- $y \in \mathbb{C}^r$: the received signal,
- $x \in \mathbb{C}^t$: the transmitted signal,
- $n \in \mathbb{C}^r$: the noise,
- $H \in \mathbb{C}^{r \times t}$: the channel gain matrix.

The elements of H are complex Gaussian with zero mean, independent real and imaginary parts, each with variance $1/2$.

SISO Channel with Log-normal shadowing

Log-normal Channel Model

$$y = hx + n$$

where

- $y \in \mathbb{C}$: the received signal,
- $x \in \mathbb{C}$: the transmitted signal,
- $n \in \mathbb{C}$: the noise,
- $h \in \mathbb{C}$: the channel gain.

$$f_h(t) = \frac{\xi}{\sqrt{2\pi}\sigma t} \exp\left(-\frac{(\xi \log(t) - \mu)^2}{2\sigma^2}\right)$$

with μ and σ being the mean and the variance of h and $\xi = \frac{10}{\log(10)}$.

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Motivation

- **What it is:** A mathematical tool consisting in letting a certain parameter (here SNR) go to extreme values (0 or $+\infty$).

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- **What it is used for:** Description of limiting behaviors, in our case channel capacity in high/low SNR regimes.
- **How it is done:** By letting SNR $\to 0/+\infty$ in exact capacity expression (excap) and deduce simple asymptotic capacity expression (ascap) such that

$$\text{excap}(\text{SNR}) \approx \text{ascap}(\text{SNR}) \iff \lim_{\text{SNR}\to 0/+\infty} \frac{excap(\text{SNR})}{ascap(\text{SNR})} = 1$$

Example 1: MIMO Channel with no CSIT

General expression

$$C^1 = \mathbb{E}_H \left[\log\det\left(I_r + \frac{\text{SNR}}{t} H H^\dagger \right) \right]$$

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Low SNR asymptotic expression

$$C_0^1 = r \,\mathrm{SNR}$$

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High SNR asymptotic expression

$$C_\infty^1 = m \log(\text{SNR})$$

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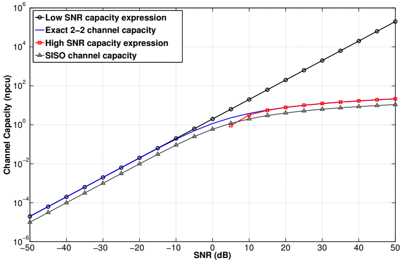

Figure : 2-2 MIMO channel capacity with perfect CSI-R in nats per channel use (npcu) versus SNR in dB.

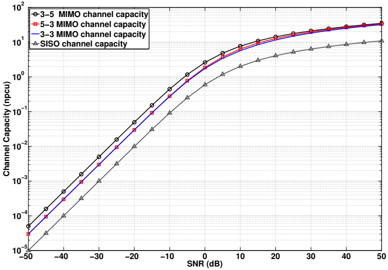

Figure : MIMO channels capacities with perfect CSI-R in nats per channel use (npcu) versus SNR in dB.

Example 2: MIMO Channel with full CSI at high SNR

General expression:

$$C^2 = m\mathbb{E}_\lambda \left[\log\left(\mu\lambda\right)^+\right]$$

where μ satisfies

$$\mathsf{SNR} = m\mathbb{E}_\lambda \left(\mu - \frac{1}{\lambda}\right)^+$$

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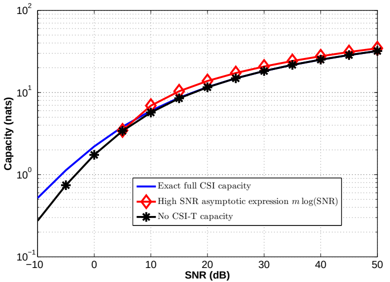

Figure : 3-3 MIMO channel capacity with full CSI in nats per channel use (npcu) versus SNR in dB.

Outline

General Expression

MIMO channel capacity under Rayleigh fading with full CSI:

$$C^2 = \max_{Q:\mathrm{Tr}[Q]\leq P_{avg}} \mathbb{E}_H \left[\log\det\left(I_r + \frac{1}{N_0} HQH^\dagger\right)\right]$$

which is equivalent to

$$C^2 = m\mathbb{E}_\lambda \left[\log\left(\mu\lambda\right)^+\right]$$

where μ satisfies

$$\mathrm{SNR} = m\mathbb{E}_\lambda \left(\mu - \frac{1}{\lambda}\right)^+$$

Low SNR Asymptotic Expression

Full CSI MIMO Capacity at Low SNR

$$
\begin{aligned}
C_0^2 &\approx \begin{cases} -\alpha \ \mathsf{SNR}\ W_0\left((\mathsf{SNR})^{\frac{1}{\alpha}}\right) & \text{if } \alpha < 0, \\ -\mathsf{SNR}\log(\mathsf{SNR}) & \text{if } \alpha = 0, \\ -\alpha \ \mathsf{SNR}\ W_{-1}\left(-(\mathsf{SNR})^{\frac{1}{\alpha}}\right) & \text{if } \alpha > 0, \end{cases} \\
&\approx \mathsf{SNR}\log(1/\mathsf{SNR})
\end{aligned}
$$

where $\alpha = n + m - 4$, $W_0(.)$ and $W_{-1}(.)$ are the main and the lower branches of the Lambert-W function, respectively.

Low SNR Asymptotic Expression

Full CSI MIMO Capacity at Low SNR

$$C_0^2 \approx \begin{cases} -\alpha \text{ SNR } W_0\left((\text{SNR})^{\frac{1}{\alpha}}\right) & \text{if } \alpha < 0, \\ -\text{SNR} \log(\text{SNR}) & \text{if } \alpha = 0, \\ -\alpha \text{ SNR } W_{-1}\left(-(\text{SNR})^{\frac{1}{\alpha}}\right) & \text{if } \alpha > 0, \end{cases}$$
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where $\alpha = n + m - 4$, $W_0(.)$ and $W_{-1}(.)$ are the main and the lower branches of the Lambert-W function, respectively.

No CSI-T Low SNR expression: $C_0^1 = r$ SNR

Illustrations

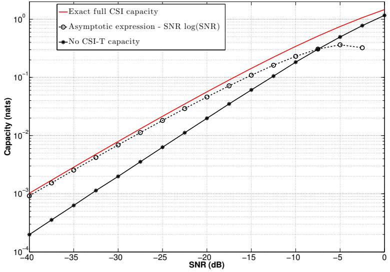

Figure : 2 transmit and 2 receive antennas channel capacity at Low-SNR in nats per channel use (npcu) versus SNR in dB.

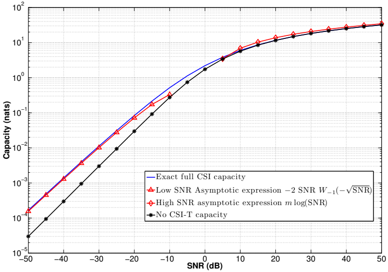

Figure : 3 transmit and 3 receive antennas channel capacity with full CSI in nats per channel use (npcu) versus SNR in dB.

On-Off scheme in the full CSI case

Power Profile

$$P(\lambda_{max}) = \begin{cases} P_0 & \text{if } \lambda_{max} \geq \tau \\ 0 & \text{otherwise} \end{cases}$$

where P_0 satisfies the average power constraint

$$P_0 = \frac{\text{SNR}}{1 - F_{\lambda_{max}}(\tau)}$$

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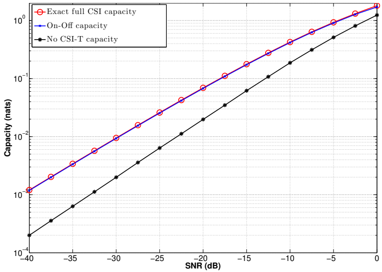

Figure : (3-2) MIMO channel capacity with full CSI and On-Off achievable rate in nats per channel use (npcu) versus SNR in dB.

Outline

System Model

Channel gain matrix

$$H = \hat{H} + \tilde{H}$$

where $\tilde{H}$ is the error matrix, and $\hat{H}$ is the estimated channel matrix. So

- $\tilde{H}_{i,j} \sim \mathcal{CN}(0, e)$
- $\hat{H}_{i,j} \sim \mathcal{CN}(0, 1 - e)$

Asymptotic Expression

Noisy CSI-T MIMO Capacity at Low SNR

$$C_0^\alpha \approx \begin{cases} -\alpha(1-e)\ \mathrm{SNR}\ W_0\left(((1-e)\mathrm{SNR})^{\frac{1}{\alpha}}\right) & \text{if } \alpha < 0, \\ -(1-e)\ \mathrm{SNR}\ \log((1-e)\mathrm{SNR}) & \text{if } \alpha = 0, \\ -\alpha(1-e)\ \mathrm{SNR}\ W_{-1}\left(-((1-e)\mathrm{SNR})^{\frac{1}{\alpha}}\right) & \text{if } \alpha > 0, \end{cases}$$

$$\approx -(1-\alpha)\ \mathrm{SNR}\ \log((1-\alpha)\mathrm{SNR})$$

where $\alpha = n + m - 4$,
e is the estimation error variance,
$W_0(.)$ and $W_{-1}(.)$ are the main and lower branches of the Lambert-W function respectively.

Illustrations

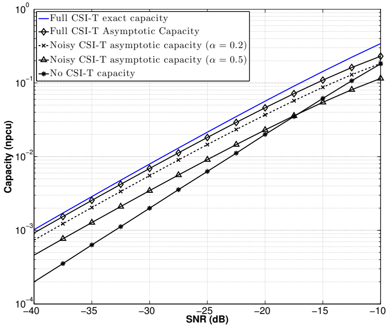

Figure : 2 transmit and 2 receive antennas channel capacity with noisy CSI-T at Low-SNR in nats per channel use (npcu) versus SNR in dB.

Outline

Lognormal Channel Capacity with full CSI

General Expression

$$C = \mathop{\mathbb{E}}_{h^2}\left[\log\left(\frac{t}{\lambda}\right)^{+}\right]$$

where λ is the water-filling level chosen to meet the power constraint

$$P_{avg} = \mathop{\mathbb{E}}_{h^2}\left[\left(\frac{1}{\lambda} - \frac{1}{t}\right)^{+}\right]$$

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Low SNR Asymptotic Expression

$$C_{ln} \approx e^{\frac{\sigma}{\xi}\sqrt{-\log(\mathrm{SNR}^2)}}\ \mathrm{SNR}$$

Illustration

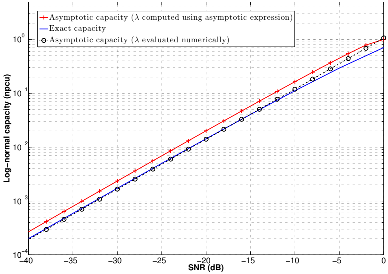

Figure : Log-normal channel capacity at Low-SNR for 0dB mean and 1 variance in nats per channel use (npcu) versus SNR in dB.

On-Off scheme in Log-normal channel

Power Profile

$$P(h) = \begin{cases} P_0 & \text{if } h \geq \lambda \\ 0 & \text{otherwise} \end{cases}$$

where P_0 satisfies the average power constraint

$$P_0 = \frac{\text{SNR}}{1 - F_h(\lambda)}$$

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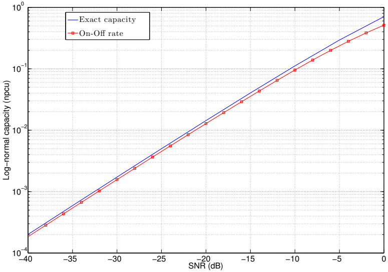

Figure : Lognormal channel capacity with full CSI and On-Off achievable rate for $\mu = 0$dB, $\sigma^2 = 1$ in nats per channel use (npcu) versus SNR in dB.

Conclusion

Contributions

- Low SNR expression of the Capacity of Full CSI MIMO Rayleigh Channel
- Low SNR expression of the Capacity of Noisy CSI-T MIMO Rayleigh Channel
- Low SNR expression of the Capacity of Log-normal shadowed Channel
- Design of On-Off transmission schemes

Conclusion

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- Low SNR expression of the Capacity of Full CSI MIMO Rayleigh Channel
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- Low SNR expression of the Capacity of Log-normal shadowed Channel
- Design of On-Off transmission schemes

Possible Extensions

- Different fading distributions
- Non ergodic capacity
- Other performance indicators (e.g. Outage Probability)

Publications

- Abdoulaye Tall, Zouheir Rezki and Mohamed-Slim Alouini, "MIMO Channel Capacity with Full CSI at Low SNR", *Accepted for publication in IEEE Wireless Communication Letters*, Jun. 2012.

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- Abdoulaye Tall, Zouheir Rezki and Mohamed-Slim Alouini, "Log-normal channel capacity characterization at low SNR with full CSI", *under preparation*.

Thank you for your attention!

Your questions are welcome.

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