MIMO Channel Capacity with Full CSI at Low SNR

Abdoulaye Tall, Zouheir Rezki, and Mohamed-Slim Alouini

***Abstract*—In this paper, we characterize the ergodic capacity of Multiple Input Multiple Output (MIMO) Rayleigh fading channels with full channel state information (CSI) at both the transmitter (CSI-T) and the receiver (CSI-R) at asymptotically low signal-to-noise ratio (SNR). A simple analytical expression of the capacity is derived for any number of transmit and receive antennas. This characterization clearly shows the substantial gain in terms of capacity over the no CSI-T case and gives a good insight on the effect of the number of antennas used. In addition, an On-Off transmission scheme is proposed and is shown to be asymptotically capacity-achieving.**

***Index Terms*—Channel capacity, MIMO, Low SNR, Full CSI, On-Off scheme**

I. INTRODUCTION

Available channel state information (CSI) at the transmitter has a considerable impact on multiple-input multiple-output (MIMO) channels capacity [1]. Such influence is negligible at high signal-to-noise ratio (SNR) [2], but for low SNR the ratio between capacity with and without CSI at the transmitter goes to infinity when the SNR goes to zero [3]. Since many systems now operate at low SNR either due to a large available bandwidth, like in wideband communications [3], or because of their own nature and their intrinsic purposes like in sensor networks [4], [5]; it is interesting to investigate how this capacity evolves with CSI at the transmitter at low SNR. A lot of work has already been done on this topic. For example in [6], spatially correlated Rayleigh and Rician fading are considered, but with no CSI at the transmitter. In [7] the full CSI case has been investigated under Rician fading. Also in [8] a low SNR expression of the MIMO capacity is derived assuming statistical CSI at the transmitter and Rician or double-scattering fading. Recently the single-input single-output (SISO) case has been investigated in [9], the capacity has been shown to scale as $\text{SNR}\log(1/\text{SNR})$ in the low SNR regime for the Rayleigh fading case. Motivated by the last result, we are interested in looking at the capacity behavior at low SNR for the general MIMO case.

Our contributions are as follows:

- We derive a low-SNR closed-form expression of a Rayleigh fading MIMO channel capacity with full CSI at the transmitter in terms of the Lambert-W function.
- We then show that at asymptotically low SNR, this capacity can be further simplified as $\text{SNR}\log(1/\text{SNR})$; suggesting that at asymptotically low-SNR, the effect of multiple antennas vanishes.
- We construct an on-off transmission scheme that is capacity-achieving at asymptotically low-SNR with only one bit feedback.

II. PROBLEM FORMULATION

Let us consider a MIMO Gaussian channel that undergoes a Rayleigh fading. We denote by t the number of transmit antennas and r the number of receive antennas. The input-output of this channel is described by:

$$y = Hx + n; \tag{1}$$

where $H \in \mathbb{C}^{r\times t}$ is the channel matrix, $x \in \mathbb{C}^t$ its input, $y \in \mathbb{C}^r$ its output and n the Gaussian noise. H is then a random complex matrix of independent and identically distributed entries, and each of its entries is assumed to be zero-mean, gaussian with independent real and imaginary parts, each part with variance 1/2. In (1), n is a zero-mean complex Gaussian noise with independent real and imaginary parts, and $\mathbb{E}(\text{nn}^\dagger) = I_r$. Let us denote by $m = \min(r,t)$ and $n = \max(r,t)$. The transmitted signal x is constrained in its total power by P_{avg} :

$$\mathbb{E}(x^\dagger x) \leq P_{avg}. \tag{2}$$

In regard of the above normalization of the noise and the channel matrix, P_{avg} will be designated as SNR. In this configuration, the capacity is given by [10] as

$$C = \mathbb{E}_H\left[\sum_{i=1}^{m}((\log(\mu\lambda_i))^+)\right], \tag{3}$$

where $\lambda_i, i = 1\cdots m$ are the eigenvalues of the matrix $HH^\dagger$ and μ is chosen via the water-filling algorithm to meet the power constraint with equality, i.e.

$$\text{SNR} = \mathbb{E}_H\left[\sum_{i=1}^{m}\left(\left(\mu - \frac{1}{\lambda_i}\right)^+\right)\right]. \tag{4}$$

While the capacity in (3) is easy to evaluate numerically, it is a priori not clear how does it scale with SNR especially at asymptotically low-SNR due to the Lagrange multiplier μ involved in (3). Next, we characterize this capacity at low SNR using an asymptotic analysis.

In the following, we consider aymptotically low SNR and say that $f \approx g$ if and only if $\lim_{\text{SNR}\to 0}\frac{f(\text{SNR})}{g(\text{SNR})} = 1$.

III. FULL CSI MIMO CAPACITY AT LOW SNR

Our main result is stated in the following Theorem.

Theorem. *The capacity of a MIMO channel undergoing Rayleigh fading as described in* (1) *with perfect CSI-T and CSI-R in the low SNR regime is given by*

$$C \approx \begin{cases} -\alpha \; \mathit{SNR} \; W_0\left((\mathit{SNR})^{\frac{1}{\alpha}}\right) & \mathit{if} \; \alpha < 0, \\ -\mathit{SNR} \log(\mathit{SNR}) & \mathit{if} \; \alpha = 0, \\ -\alpha \; \mathit{SNR} \; W_{-1}\left(-(\mathit{SNR})^{\frac{1}{\alpha}}\right) & \mathit{if} \; \alpha > 0, \end{cases} \tag{5}$$

$$\approx \mathit{SNR} \log(1/\mathit{SNR}) \tag{6}$$

where $\alpha = n + m - 4$, $W_0(.)$ and $W_{-1}(.)$ are the main and the lower branches of the Lambert-W function, respectively.

Proof: First let us find the water-filling level μ that corresponds to optimal power allocation. From (4) we have

$$\mathrm{SNR} = m \; \mathbb{E}_\lambda \left(\left(\mu - \frac{1}{\lambda} \right)^+ \right), \tag{7}$$

The probability density function (PDF) of λ is given by [11, Eq. (42)];

$$p_\lambda(\lambda) = \frac{1}{m} \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{l=0}^{2j} C(n,m,i,j,l) \lambda^{n-m+l} e^{-\lambda} \tag{8}$$

where $C(n,m,i,j,l)$ is given by

$$C(n,m,i,j,l) = \frac{(-1)^l (2j)!}{2^{(2i-l)} j! l! (n-m+j)!} \binom{2i-2j}{i-j} \binom{2n-2m+2j}{2j-l} \tag{9}$$

From (7) and (8), we have

$$\mathrm{SNR} = \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{l=0}^{2j} C(n,m,i,j,l) \int_{\frac{1}{\mu}}^{\infty} \left(\mu - \frac{1}{\lambda} \right) \lambda^{n-m+l} e^{-\lambda} \, d\lambda,$$

With the use of [12, eq. (8.350.2) p. 949] which defines the upper incomplete Gamma function $\Gamma(.,.)$, we have

$$\int_{\frac{1}{\mu}}^{\infty} \left(\mu - \frac{1}{\lambda} \right) \lambda^{n-m+l} e^{-\lambda} \, d\lambda = \mu \Gamma\left(n-m+l+1, \frac{1}{\mu} \right) - \Gamma\left(n-m+l, \frac{1}{\mu} \right). \tag{10}$$

$\Gamma(.,.)$ can be rewritten using [12, eq. (8.352.2) p. 949] as $\Gamma(k,\alpha) = (k-1)! e^{-\alpha} \sum_{j=0}^{k-1} \frac{\alpha^j}{j!}$. As SNR $\to 0$, we can find that the water level μ also goes to 0 using Lemma 1 in [13]. Thus we can say that

$$\mu^p \ll \mu^q \;\; \forall p,q \in \mathbb{N} \text{ such as } p > q. \tag{11}$$

Using (11) in (10), we obtain

$$\int_{\frac{1}{\mu}}^{\infty} \left(\mu - \frac{1}{\lambda} \right) \lambda^{n-m+l} e^{-\lambda} \, d\lambda \approx \mu^{-(n-m+l-2)} e^{-\frac{1}{\mu}}$$

Finally, we get

$$\mathrm{SNR} \approx \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{l=0}^{2j} C(n,m,i,j,l) \mu^{-(n-m+l-2)} e^{-\frac{1}{\mu}}. \tag{12}$$

Using the same considerations as in (11), we can restrict this expression to the term with highest value of l which is $l = 2(m-1)$. For this particular value

$$C(n,m,i,j,l) = C(n,m,m-1,m-1,2(m-1)) = \frac{1}{(m-1)!(n-1)!}.$$

As such we end up with

$$\mathrm{SNR} \approx \frac{1}{(m-1)!(n-1)!} \mu^{-(n+m-4)} e^{-\frac{1}{\mu}}. \tag{13}$$

This equation can be solved by transforming it into an equation of the type $y = xe^x$ which is solved in terms of the Lambert-W function. More specifically we have three cases:

Case 1: $n+m-4 < 0$: In this case the solution is given by the main branch of Lambert W function $W_0(.)$ as

$$\frac{1}{\mu} \approx -(n+m-4) W_0 \left(-\frac{(\mathrm{SNR}(m-1)!(n-1)!)^{\frac{1}{n+m-4}}}{n+m-4} \right). \tag{14}$$

Case 2: $n+m-4 = 0$: This is the simplest case where in fact $(m,n) \in \{(2,2),(1,3)\}$. In this case μ is simply given by

$$\frac{1}{\mu} \approx -\log(\mathrm{SNR}(n-1)!). \tag{15}$$

Case 3: $n+m-4 > 0$: In this case, the lower branch of the lambert W function is used since the argument given to that function would be negative, and our solution must rationally be converging towards 0 when the power do so. Thus in this case we get

$$\frac{1}{\mu} \approx (n+m-4) W_{-1} \left(-\frac{(\mathrm{SNR}(m-1)!(n-1)!)^{\frac{1}{n+m-4}}}{n+m-4} \right). \tag{16}$$

However this solution is meaningless when the argument of $W_{-1}(.)$ function is less than $-\frac{1}{e}$. As such our average power must be beneath a certain value for (16) to be valid, and that value is given by

$$\mathrm{SNR} \leq \frac{1}{(m-1)!(n-1)!} \left(\frac{n+m-4}{e} \right)^{(n+m-4)}. \tag{17}$$

Using the fact that

$$\lim_{x \to \infty} \frac{W_0(\beta x)}{W_0(x)} = 1 \text{ and } \lim_{x \to 0^-} \frac{W_{-1}(\beta x)}{W_{-1}(x)} = 1 \;\forall \beta > 0, \tag{18}$$

as shown in [13, Eq. (16),(17)], we can further simplify (14) and (16). A more simplified solution of (13) can be obtained by applying the log function on both sides of (13) and recalling that when SNR $\to 0$ then $\mu \to 0$. We can also neglect $\log((m-1)!(n-1)!)$ besides $\log(\mathrm{SNR})$ at asymptotically low SNR for fixed m and n. In this case, we get

$$\mu \approx -\frac{1}{\log(\mathrm{SNR})}. \tag{19}$$

The water level μ being characterized, let us compute the capacity. From (3), we have

$$C = m \; \mathbb{E}((\log(\mu\lambda))^+). \tag{20}$$

Using again the PDF of the eigenvalues given in (8), we obtain

$$C = \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{l=0}^{2j} C(n,m,i,j,l) \int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda)) \lambda^{n-m+l} e^{-\lambda} \, d\lambda.$$

Using [12, Eq. (8.350.2) p. 949], we can write

$$\int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda))\lambda^{n-m+l}e^{-\lambda}\,d\lambda = \log(\mu)\Gamma\left(n-m+l+1,\frac{1}{\mu}\right) + \int_{\frac{1}{\mu}}^{\infty} \log(\lambda)\lambda^{n-m+l}e^{-\lambda}\,d\lambda. \tag{21}$$

Letting $t=\mu\lambda$ in the second term of (21), we get

$$\int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda))\lambda^{n-m+l}e^{-\lambda}\,d\lambda = \log(\mu)\Gamma\left(n-m+l+1,\frac{1}{\mu}\right) + \log\left(\frac{1}{\mu}\right)\Gamma\left(n-m+l+1,\frac{1}{\mu}\right) + \left(\frac{1}{\mu}\right)^{n-m+l+1}\int_{\frac{1}{\mu}}^{\infty} \log(t)t^{n-m+l}e^{-\frac{t}{\mu}}\,dt. \tag{22}$$

From [14, Eq. (64)], we know that

$$\int_{\frac{1}{\mu}}^{\infty} \log(t)t^{n-m+l}e^{-\frac{t}{\mu}}\,dt = \frac{(n-m+l)!}{\left(\frac{1}{\mu}\right)^{n-m+l+1}}\sum_{k=0}^{n-m+l}\frac{\Gamma(k,\frac{1}{\mu})}{k!} \tag{23}$$

In addition to that, using similar techniques as in (11) for (22), we can write

$$\int_{\frac{1}{\mu}}^{\infty} (\log(\mu\lambda))\lambda^{n-m+l}e^{-\lambda}\,d\lambda \approx \left(\frac{1}{\mu}\right)^{n-m+l+1} e^{-\frac{1}{\mu}}.$$

Thus, the capacity at low SNR can be written as

$$C \approx \sum_{i=0}^{m-1}\sum_{j=0}^{i}\sum_{l=0}^{2j} C(n,m,i,j,l)\left(\frac{1}{\mu}\right)^{n-m+l+1} e^{-\frac{1}{\mu}}. \tag{24}$$

In (24) we recognize the expression of the SNR given in (12). Thus we can write

$$C \approx \frac{\text{SNR}}{\mu}. \tag{25}$$

Combining (14), (15), (16), and (18) along with (25) gives (5), whereas (6) follows from (19) and (25). ■

IV. On-Off transmission scheme

In this section, we propose a transmission scheme that is asymptotically (at low-SNR) capacity-achieving. Intuitively since the SNR is very low, one should remain silent and send opportunistically when the channel is very good using a Gaussian codebook and a fixed transmit power. Also, since the available SNR is low, one can reasonably try to exploit the strongest channel eigenmode and transmit along the largest eigenvalue direction. Clearly, this can be done by appropriately activating only one antenna at each end and turning off the others. More specifically, the on-off power control scheme is

$$P(\lambda_{max}) = \begin{cases} P_0 & \text{if } \lambda_{max} \geq \tau \\ 0 & \text{otherwise} \end{cases}$$

where τ is a threshold and P_0 satisfies the average power constraint

$$P_0 = \frac{\text{SNR}}{1-F_{\lambda_{max}}(\tau)}$$

The cumulative distribution function of the largest eigenvalue of a Wishart matrix is given in [15, eq. (6)] by

$$F_{\lambda_{max}}(x) = \frac{det(\Psi(x))}{\prod_{k=1}^{m}\Gamma(n-k+1)\Gamma(m-k+1)}$$

where $\Psi(x)$ is a mxm matrix defined as follows:

$$\Psi_{(i,j)}(x) = \gamma(n-m+i+j-1,x); i,j=1,...,m$$

where $\gamma(.)$ is the lower incomplete gamma function [12, Eq. (8.350.1) p. 949]. Taking advantage of our framework, we choose τ to be the inverse of the water-filling level μ which is the solution of (13), its low-SNR expression is given by (14) , (15) or (16) depending on the values of m and n. Therefore, the achievable rate, say R, of this on-off scheme is given by

$$\begin{aligned} R &= \mathbb{E}_{\lambda_{max}}\left(\log(1+P(\lambda_{max})\lambda_{max})\right) \\ &= \int_{\tau}^{\infty}\log(1+P_0\lambda)p_{\lambda_{max}}(\lambda)d\lambda \qquad (26) \\ &\geq \log(1+P_0\tau)\int_{\tau}^{\infty}p_{\lambda_{max}}(\lambda)d\lambda \qquad (27) \end{aligned}$$

Now let us note that $F_{\lambda_{max}}(\tau) \leq F_{\lambda}(\tau)$. Thus

$$P_0\tau = \frac{\text{SNR}\tau}{1-F_{\lambda_{max}}(\tau)} \leq \frac{\text{SNR}\tau}{1-F_{\lambda}(\tau)}.$$

From (13) and considering the value of τ we have chosen, it can be easily verified that SNR$\tau \approx K_1\tau^{n+m-3}e^{-\tau}$. Also using the PDF of λ in (8) we can show that $1-F_{\lambda}(\tau) \approx K_2\tau^{n+m-2}e^{-\tau}$ for $\tau \to \infty$. K_1 and K_2 are positive real constants. So, it becomes clear that

$$\lim_{\tau\to\infty} P_0\tau = 0 \tag{28}$$

Then combining (27) and (28), and recalling that $\log(1+x) \approx x$; the proposed on-off achievable rate can be lower-bounded by

$$R \geq P_0\tau(1-F_{\lambda_{max}}(\lambda)) = \text{SNR } \tau \tag{29}$$

The right hand side (RHS) of (29) is nothing but the asymptotic capacity expression given by (25) which ensures that the proposed on-off scheme is asymptotically capacity-achieving. In fact this on-off scheme is still capacity achieving if we choose any of the eigenvalues (different distribution than that of the maximum eigenvalue) at each realisation of the channel and the proof is similar. This means that we can use only one antenna at both sides of the channel and still achieve this capacity. So for this on-off scheme, we only need one bit feedback from the receiver stating whether we transmit or not.

V. Numerical Results

In Fig. 1, Fig. 2, and Fig. 3; the exact capacity is computed using standard root-finding algorithms to evaluate the water-filling level μ and then using numerical integration. The no CSI-T capacity is obtained by Monte-Carlo simulations. The tightness of the expressions given in (5) and (6) is clearly visible in Fig. 1, Fig. 2, and Fig. 3. The growing gap between the full CSI capacity and the no-CSI capacity when the SNR converges towards zero stress on the fact that CSI at the

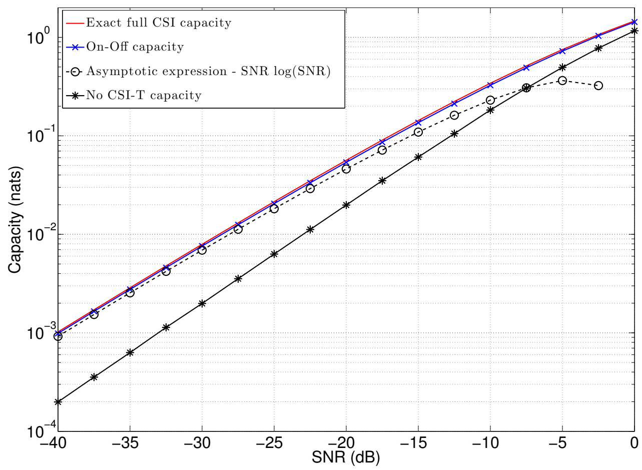

Fig. 1. 2 transmit and 2 receive antennas channel capacity at Low-SNR in nats per channel use (npcu) versus SNR in dB.

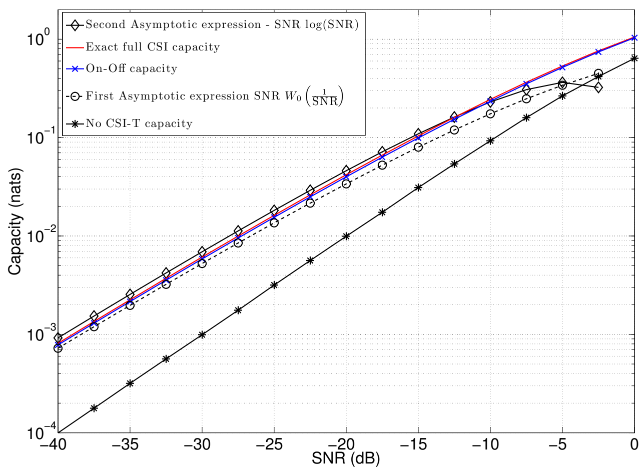

Fig. 2. 2 transmit and 1 receive antennas channel capacity with full CSI at Low-SNR in nats per channel use (npcu) versus SNR in dB.

transmitter affects considerably the capacity at low SNR. Fig. 1 depicts the particular case of $n = m = 2$ in which the two expressions (5) and (6) become identical. In Fig. 2, the first asymptotic expression uses the main branch of the Lambert function since $n + m < 4$; we can also note that in this case the second asymtotic expression approaches the exact capacity from above. But in Fig. 3, the lower branch is used instead because $n + m > 4$ and the second asymptotic expression approaches the exact capacity from below. The achievable rate of the proposed on-off scheme curve has been obtained by evaluating (26), where the PDF of λ_{max} has been borrowed from [16, Eq. (23)]. In Fig. 1, Fig. 2 and Fig. 3, we can see that the on-off rate overlaps always with the exact capacity, suggesting that this scheme is very appealing from a practical point of view, even for not so low values of the SNR.

VI. Conclusion

In this paper, we derived a simple analytical expression for the capacity of MIMO channel undergoing Rayleigh fading with full CSI-T in the low SNR regime. We also showed that at aymptotically low SNR, there is no need for multiple antennas.

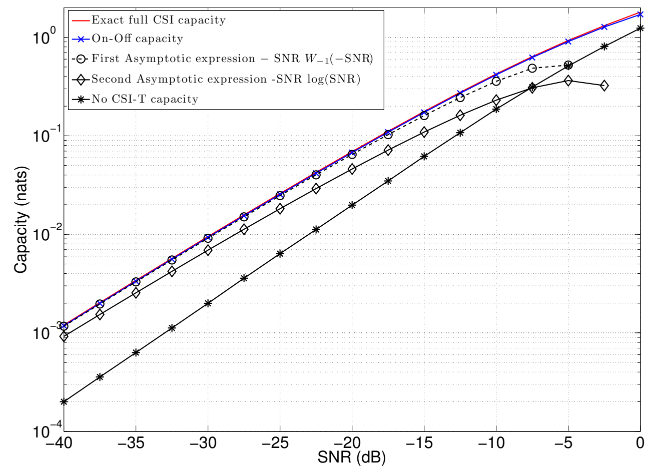

Fig. 3. 3 transmit and 2 receive antennas channel capacity at Low-SNR in nats per channel use (npcu) versus SNR in dB.

We even constructed a simple on-off scheme which only needs 1-bit feedback from the receiver, only one antenna on both sides and which is asymptotically capacity-achieving.

References

[1] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," *IEEE J. Select. Areas Commun.*, vol. 21, no. 5, pp. 684–702, May 2003.

[2] G. J. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Personal Communications*, vol. 6, pp. 311–335, Mar. 1998.

[3] S. Verdu, "Spectral efficiency in the wideband regime," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1319–1343, Jun. 2002.

[4] J.-F. Chamberland and V. Veeravalli, "Asymptotic results for decentralized detection in power constrained wireless sensor networks," *IEEE J. Sel. Areas Commun.*, vol. 22, no. 6, pp. 1007–1015, Aug. 2004.

[5] S. Jayaweera, "Virtual MIMO-based cooperative communication for energy-constrained wireless sensor networks," *IEEE Trans. Wireless Commun.*, vol. 5, no. 5, pp. 984–989, may 2006.

[6] A. Lozano, A. Tulino, and S. Verdu, "Multiple-antenna capacity in the low-power regime," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2527–2544, Oct. 2003.

[7] Z. Rezki and M.-S. Alouini, "On the capacity of Rician fading channels with full channel state information at low SNR," in *Proc. IEEE Int. Conf. on Commun., (ICC '12), Ottawa, Canada*, Jun. 2012, pp. 1–5.

[8] X. Li, S. Jin, M. McKay, X. Gao, and K. kit Wong, "Capacity of MIMO-MAC with transmit channel knowledge in the low SNR regime," *IEEE Trans. Wireless Commun.*, vol. 9, no. 3, pp. 926 –931, Mar. 2010.

[9] S. Borade and L. Zheng, "Wideband fading channels with feedback," *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6058–6065, Dec. 2010.

[10] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Transactions on Telecommunications*, vol. 10, pp. 585–595, Nov. 1999.

[11] H. Shin and J. H. Lee, "Capacity of multiple-antenna fading channels: spatial fading correlation, double scattering, and keyhole," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2636–2647, Oct. 2003.

[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. San Diego, CA: Academic, 1994.

[13] Z. Rezki and M. Alouini, "On the capacity of Nakagami-m fading channels with full channel state information at low SNR," *IEEE Commun. Lett.*, To appear (available on IEEEXplore).

[14] M.-S. Alouini and A. Goldsmith, "Capacity of Rayleigh fading channels under different adaptive transmission and diversity-combining techniques," *IEEE Trans. Veh. Technol.*, vol. 48, no. 4, pp. 1165–1181, Jul. 1999.

[15] M. Kang and M.-S. Alouini, "A comparative study on the performance of MIMO MRC systems with and without cochannel interference," *IEEE Trans. Commun.*, vol. 52, no. 8, pp. 1417–1425, Aug. 2004.

[16] P. Dighe, R. Mallik, and S. Jamuar, "Analysis of transmit-receive diversity in Rayleigh fading," *IEEE Trans. Commun.*, vol. 51, no. 4, pp. 694–703, April 2003.